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Optimal expressions for solution matrices of single – delay differential systems.

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ABSTRACT: *This paper derived optimal expressions for solution matrices of single – delay autonomous linear differential systems on any given interval of length equal to the delay h for non –negative time periods. The formulation and the development of the theorem exploited an earlier work Ukwu (2013b) on the interval [0, 4h]. The proofs were achieved using ingenious combinations of summation notations, multiple product notations and integrals, as well as the method of steps to obtain these matrices on successive intervals of length equal to the delay h. This theorem globally extends the time scope of applications of these matrices to the solutions of initial function problems, rank conditions for controllability and cores of targets, constructions of controllability Grammians and admissible controls for transfers of points associated with controllability problems.*

KEYWORDS- *Delay, Matrices, Solution, Structure, Systems.*

I. INTRODUCTION

The qualitative approach to the controllability of functional differential control systems have been areas of active research for the past fifty years among control theorists and applied mathematicians in general. This circumvents the severe difficulties associated with the search for and computations of solutions of such systems.

Unfortunately computations of solutions cannot be wished away in the tracking of trajectories and many practical applications. Literature on state space approach to control studies is replete with variation of constants formulas, which incorporate the solution matrices of the free part of the systems. See Chukwu (1992), Gabsov and Kirillova (1976), Hale (1977), Manitius (1978), Tadmore (1984), and Ukwu (1987, 1992,1996). Regrettably no author has made any attempt to obtain general expressions for such solution matrices involving the delay h.

The usual approach is to start from the interval $[0, h]$ and compute the solution matrices and solutions for given problem instances and then use the method of steps to extend these to the intervals $[kh, (k+1)h]$, for positive integral k, not exceeding 2, for the most part. Such approach is rather restrictive and doomed to failure in terms of structure for arbitrary k. In other words such approach fails to address the issue of the structure and computing complexity of solution matrices and solutions quite vital for real-world applications. The need to address such short-comings has become imperative; this is the major contribution of this paper, with its wide-ranging implications for extensions to double-delay and neutral systems and holistic approach to controllability studies.

II. PRELIMINARIES

We consider the class of double-delay differential systems:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h),\ t \in \mathbf{R} \tag{1}$$

where A_0 and A_1 are $n \times n$ constant matrices and $x(.)$ is an n -dimensional column vector function.

Let $Y_{k-i}(t-ih)$ be a solution matrix of (1), on the interval $J_{k-i} = [(k-i)h, (k+1-i)h],\ k \in \{0,1,\cdots\}, i \in \{0,1\}$, where;

$$Y(t) = \begin{cases} I_n, t = 0 \\ 0,\ t < 0. \end{cases}$$

Note: $Y(t)$ is a generic solution matrix for any $t \in \mathbf{R}$ and I_n is the identity matrix of order n.

The solution matrices will be obtained piece – wise on successive intervals of length h.

Ukwu (2013b) obtained the following expressions for the solution matrices, $Y(t)$ on J_k, for $k \in \{0, \cdots, 3\}$:

$$
Y(t)=\begin{cases}
e^{A_0 t},\ t\in J_0; & (2)\\
e^{A_0 t}+\int_h^t e^{A_0(t-s)}A_1 e^{A_0(s-h)}ds,\ t\in J_1; & (3)\\
e^{A_0 t}+\int_h^t e^{A_0(t-s)}A_1 e^{A_0(s-h)}ds+\int_{2h}^t\int_h^{s_2-h} e^{A_0(t-s_2)}A_1 e^{A_0(s_2-s_1-h)}A_1 e^{A_0(s_1-h)}ds_1 ds_2,\ t\in J_2; & (4)\\
e^{A_0 t}+\int_h^t e^{A_0(t-s)}A_1 e^{A_0(s-h)}ds+\int_{2h}^t\int_h^{s_2-h} e^{A_0(t-s_2)}A_1 e^{A_0(s_2-s_1-h)}A_1 e^{A_0(s_1-h)}ds_1 ds_2 & \\
\quad+\int_{3h}^t\int_{2h}^{s_3-h}\int_h^{s_2-h} e^{A_0(t-s_3)}A_1 e^{A_0(s_3-h-s_2)}A_1 e^{A_0(s_2-h-s_1)}A_1 e^{A_0(s_1-h)}ds_1 ds_2 ds_3,\ t\in J_3 & (5)
\end{cases}
$$

He also interrogated some topological dispositions of the solution matrices and deduced that the solution matrices are continuous on the interval [0,4h] but not analytic $t\in\{0,\ h,\ 2h,\ 3h\}$. These results are consistent with the existing qualitative theory on $Y(t)$. See Chukwu (1992), Hale (1977), Tadmore (1984) and Ukwu (1987, 1996). See also analytic function (2010) and Chidume (2007) for discussions on analytical functions and topology.

The objective of this paper is to formulate and prove a theorem on the general expression for $Y(t)$ on J_k, for $k\in\{0,\ 1,\ \cdots\}$, by appropriating the above expression for $Y(t)$.

1. **Theorem:** Ukwu-Garba's Solution Matrix Formula for Autonomous, Single – Delay Linear Systems:

$$
Y(t)=\begin{cases}
e^{A_0 t},\ t\in J_0; & (6)\\
e^{A_0 t}+\int_h^t e^{A_0(t-s)}A_1 e^{A_0(s-h)}ds,\ t\in J_1; & (7)\\
e^{A_0 t}+\int_h^t e^{A_0(t-s)}A_1 e^{A_0(s-h)}ds & \\
+\left[\sum_{j=2}^{k}\int_{jh}^{t} e^{A_0(t-s_j)}\prod_{i\in\{j,\,j-1,\cdots,2\}}\int_{(i-1)h}^{s_i-h} A_1 e^{A_0(s_i-h-s_{i-1})}\right]A_1 e^{A_0(s_1-h)}\prod_{\lambda=1}^{j} ds_\lambda,\ t\in J_k,\ k\ge 2 & (8)
\end{cases}
$$

Proof

The expressions (2) and (3) prove (6) and (7) respectively. If $j=2$, in (8) we obtain:

$$
Y(t)=e^{A_0 t}+\int_h^t e^{A_0(t-s)}A_1 e^{A_0(s-h)}ds+\int_{2h}^t e^{A_0(t-s_j)}\int_h^{s_2-h} A_1 e^{A_0(s_2-h-s_1)}A_1 e^{A_0(s_1-h)}ds_1 ds_2,\ t\in J_2
$$

$$
\Rightarrow\ Y(t)=e^{A_0 t}+\int_h^t e^{A_0(t-s)}A_1 e^{A_0(s-h)}ds+\int_{2h}^t\int_h^{s_2-h} e^{A_0(t-s_2)}A_1 e^{A_0(s_2-h-s_1)}A_1 e^{A_0(s_1-h)}ds_1 ds_2,\ t\in J_2 \qquad (9)
$$

The expression (9) agrees with (4); therefore the theorem is valid for $t\in J_2$. If $j=3$, in (8), we get:

$$
Y(t)=e^{A_0 t}+\int_h^t e^{A_0(t-s)}A_1 e^{A_0(s-h)}ds+\int_{2h}^t e^{A_0(t-s_2)}\int_{2h}^{s_2-h} A_1 e^{A_0(s_2-h-s_1)}A_1 e^{A_0(s_1-h)}ds_1 ds_2
$$

$$
+\int_{3h}^t e^{A_0(t-s_3)}\int_{2h}^{s_3-h} A_1 e^{A_0(s_3-h-s_2)}\int_h^{s_2-h} A_1 e^{A_0(s_2-h-s_1)}A_1 e^{A_0(s_1-h)}ds_1 ds_2 ds_3
$$

$$\Rightarrow Y(t) = e^{A_0 t} + \int_h^t e^{A_0(t-s)} A_1 e^{A_0(s-h)} ds + \int_{2h}^{t} \int_{2h}^{s_2-h} e^{A_0(t-s_2)} A_1 e^{A_0(s_2-h-s_1)} A_1 e^{A_0(s_1-h)} ds_1 ds_2$$

$$+ \int_{3h}^{t} \int_{2h}^{s_3-h} \int_{h}^{s_2-h} e^{A_0(t-s_3)} A_1 e^{A_0(s_3-h-s_2)} A_1 e^{A_0(s_2-h-s_1)} A_1 e^{A_0(s_1-h)} ds_1 ds_2 ds_3 \quad (10)$$

This result is consistent with the expression (5). Therefore the theorem is also valid for $t \in J_3$. The rest of the proof is inductive. Assume the validity of the theorem on J_p, $4 \le p \le k$ for some integer k. Then on J_{k+1} we have:

$$Y(t) = e^{A_0(t-[k+1]h)} Y([k+1]h) + \int_{(k+1)h}^{t} e^{A_0(t-s_{k+1})} A_1 Y(s_{k+1}-h) ds_{k+1} \quad (11)$$

$s_{k+1} \in J_{k+1} \Rightarrow s_{k+1} - h \in J_k \Rightarrow$ the induction hypothesis applies to $Y(s_{k+1}-h)$ and $Y([k+1]h)$.
Therefore:

$$Y([k+1]h) = e^{A_0(k+1)h} + \int_h^{(k+1)h} e^{A_0([k+1]h-s_1)} A_1 e^{A_0(s_1-h)} ds_1 \quad (12)$$

$$+ \left[\sum_{j=2}^{k} \int_{jh}^{(k+1)h} e^{A_0([k+1]h-s_j)} \prod_{i\in\{j,j-1,\cdots,2\}} \int_{(i-1)h}^{s_i-h} A_1 e^{A_0(s_i-h-s_{i-1})} \right] A_1 e^{A_0(s_1-h)} \prod_{\lambda=1}^{j} ds_\lambda, t \in J_k, k \ge 2 \quad (13)$$

$$\Rightarrow Y(t) = e^{A_0 t} + \int_h^{(k+1)h} e^{A_0(t-s_1)} A_1 e^{A_0(s_1-h)} ds_1$$

$$+ \left[\sum_{j=2}^{k} \int_{jh}^{(k+1)h} e^{A_0(t-s_j)} \prod_{i\in\{j,j-1,\cdots,2\}} \int_{(i-1)h}^{s_i-h} A_1 e^{A_0(s_i-h-s_{i-1})} \right] A_1 e^{A_0(s_1-h)} \prod_{\lambda=1}^{j} ds_\lambda \quad (14)$$

$$+ \int_{(k+1)h}^{t} e^{A_0(t-s_{k+1})} A_1 e^{A_0(s_{k+1}-h)} ds_{k+1} + \int_{(k+1)h}^{t} e^{A_0(t-s_{k+1})} A_1 \int_h^{s_{k+1}-h} A_1 e^{A_0(s_{k+1}-h-s_1)} A_1 e^{A_0(s_1-h)} ds_1 ds_{k+1} \quad (15)$$

$$+ \left[\sum_{j=2}^{k} \int_{(k+1)h}^{t} e^{A_0(t-s_{k+1})} A_1 \int_{jh}^{s_{k+1}-h} {}_1 e^{A_0(s_{k+1}-h-s_j)} \prod_{i\in\{j,j-1,\cdots,2\}} \int_{(i-1)h}^{s_i-h} A_1 e^{A_0(s_i-h-s_{i-1})} \right] A_1 e^{A_0(s_1-h)} \prod_{\lambda=1}^{j} ds_\lambda ds_{k+1} \quad (16)$$

$$\Rightarrow Y(t) = e^{A_0 t} + \int_h^{(k+1)h} e^{A_0(t-s_1)} A_1 e^{A_0(s_1-h)} ds_1 + \int_{(k+1)h}^{t} e^{A_0(t-s_{k+1})} A_1 e^{A_0(s_{k+1}-h)} ds_{k+1} \quad (17)$$

$$+ \left[\sum_{j=2}^{k} \int_{jh}^{(k+1)h} e^{A_0(t-s_j)} \prod_{i\in\{j,j-1,\cdots,2\}} \int_{(i-1)h}^{s_i-h} A_1 e^{A_0(s_i-h-s_{i-1})} \right] A_1 e^{A_0(s_1-h)} \prod_{\lambda=1}^{j} ds_\lambda \quad (18)$$

$$+ \left[\sum_{j=2}^{k} \int_{(k+1)h}^{t} e^{A_0(t-s_{k+1})} A_1 \int_{jh}^{s_{k+1}-h} {}_1 e^{A_0(s_{k+1}-h-s_j)} \prod_{i\in\{j,j-1,\cdots,2\}} \int_{(i-1)h}^{s_i-h} A_1 e^{A_0(s_i-h-s_{i-1})} \right] A_1 e^{A_0(s_1-h)} \prod_{\lambda=1}^{j} ds_\lambda ds_{k+1} \quad (19)$$

$$+ \int_{(k+1)h}^{t} e^{A_0(t-s_{k+1})} A_1 \int_h^{s_{k+1}-h} A_1 e^{A_0(s_{k+1}-h-s_1)} A_1 e^{A_0(s_1-h)} ds_1 ds_{k+1} \quad (20)$$

$$\Rightarrow Y(t) = e^{A_0 t} + \int_h^t e^{A_0(t-s_1)} A_1 e^{A_0(s_1-h)} ds_1$$

$$+ \left[\sum_{j=3}^{k+1} \int_{jh}^{(k+1)h} e^{A_0(t-s_j)} \prod_{i\in\{j,j-1,\cdots,2\}} \int_{(i-1)h}^{s_i-h} A_1 e^{A_0(s_i-h-s_{i-1})} \right] A_1 e^{A_0(s_1-h)} \prod_{\lambda=1}^{j} ds_\lambda \quad (21)$$

$$+\int_{2h}^{(k+1)h} e^{A_0(t-s_j)}A_1\int_{h}^{s_2-h} A_1 e^{A_0(s_2-h-s_1)}A_1 e^{A_0(s_1-h)}ds_1 ds_2 \qquad (22)$$

$$+\left[\sum_{j=3}^{k}\int_{(k+1)h}^{t} e^{A_0(t-s_{k+1})}A_1\int_{(j-1)h}^{s_{k+1}-h} {}_1e^{A_0(s_{k+1}-h-s_{j-1})}\prod_{i\in\{j-1,\cdots,2\}}\int_{(i-1)h}^{s_i-h} A_1 e^{A_0(s_i-h-s_{i-1})}\right]A_1 e^{A_0(s_1-h)}\prod_{\lambda=1}^{j-1} ds_\lambda ds_{k+1} \qquad (23)$$

$$+\int_{(k+1)h}^{t} e^{A_0(t-s_{k+1})}A_1\int_{h}^{s_{k+1}-h} A_1 e^{A_0(s_{k+1}-h-s_1)}A_1 e^{A_0(s_1-h)}ds_1 ds_{k+1} \qquad (24)$$

Note the use of change of variables in obtaining (23) and that $j = k + 1$ zeros out (21).
(22) and (24) add up to yield:

$$\int_{2h}^{t} e^{A_0(t-s_2)}A_1\int_{h}^{s_2-h} A_1 e^{A_0(s_2-h-s_1)}A_1 e^{A_0(s_1-h)}ds_1 ds_2 \qquad (25)$$

(23) may be rewritten in the form:

$$\left[\sum_{j=3}^{k}\int_{(k+1)h}^{t} e^{A_0(t-s_{k+1})}\prod_{i\in\{j,j-1,\cdots,2\}}\int_{(i-1)h}^{s_i-h} A_1 e^{A_0(s_i-h-s_{i-1})}\right]A_1 e^{A_0(s_1-h)}\prod_{\lambda=1}^{j} ds_\lambda \qquad (26)$$

Hence (21) and (26) add up to yield:

$$\left[\sum_{j=3}^{k+1}\int_{jh}^{t} e^{A_0(t-s_j)}\prod_{i\in\{j,j-1,\cdots,2\}}\int_{(i-1)h}^{s_i-h} A_1 e^{A_0(s_i-h-s_{i-1})}\right]A_1 e^{A_0(s_1-h)}\prod_{\lambda=1}^{j} ds_\lambda \qquad (27)$$

Add up $e^{A_0 t}+\int_h^t e^{A_0(t-s_1)}A_1 e^{A_0(s_1-h)}ds_1$,(25) and (27) to get:

$$Y(t) = e^{A_0 t}+\int_h^t e^{A_0(t-s_1)}A_1 e^{A_0(s_1-h)}ds_1$$

$$+\left[\sum_{j=2}^{k+1}\int_{jh}^{t} e^{A_0(t-s_j)}\prod_{i\in\{j,j-1,\cdots,2\}}\int_{(i-1)h}^{s_i-h} A_1 e^{A_0(s_i-h-s_{i-1})}\right]A_1 e^{A_0(s_1-h)}\prod_{\lambda=1}^{j} ds_\lambda, \text{ on } J_k. \qquad (28)$$

So, the theorem is valid on J_{k+1}, and hence valid for all $J_k, k\in\{0,1,2,\cdots\}$. This completes the proof.

3.1 Corollary 1

If $A_1 = \mathrm{diag}(b)$, then:

$$Y(t)=\begin{cases} e^{A_0 t},\ t\in J_0; & (29)\\ e^{A_0 t}+\displaystyle\sum_{i=1}^{k}\frac{b^i\ (t-ih)^i}{i!}e^{A_0(t-ih)},\ t\in J_k, k\geq 1 & (30)\end{cases}$$

Proof

The proof follows by straight-forward successive integration, noting that $A_1 = \mathrm{diag}(b) \Rightarrow A_1$ commutes with $e^{A_0(.)}$; So $Y(t)$ reduces to :

$$Y(t)=\begin{cases} e^{A_0 t},\ t\in J_0; & (31)\\ e^{A_0 t}+\int_h^t b e^{A_0(t-h)}ds = e^{A_0 t}+b(t-h)e^{A_0(t-h)},\ t\in J_1; & (32)\\ e^{A_0 t}+b(t-h)e^{A_0(t-h)} & \\ +\left[\sum_{j=2}^{k}\int_{jh}^{t} e^{A_0(t-s_j)}\prod_{i\in\{j,j-1,\cdots,2\}}\int_{(i-1)h}^{s_i-h} b e^{A_0(s_i-h-s_{i-1})}\right] b e^{A_0(s_1-h)}\prod_{\lambda=1}^{j} ds_\lambda,\ t\in J_k, k\ge 2 & (33)\end{cases}$$

The sum of the multiple integrals in (33) yields:

$$\sum_{j=2}^{k} b^j e^{A_0(t-jh)}\prod_{i\in\{j,j-1,\cdots,2\}}\int_{jh}^{t}\int_{(i-1)h}^{s_i-h} ds_1 ds_2\cdots ds_j=\sum_{j=2}^{k} b^j e^{A_0(t-jh)}\frac{(t-jh)^j}{j!}=\sum_{i=2}^{k} b^i e^{A_0(t-ih)}\frac{(t-ih)^i}{i!} \quad (34)$$

Adding the expression $e^{A_0 t}+b(t-h)e^{A_0(t-h)}$ to the expression (3.39) yields the result:

$$Y(t)=e^{A_0 t}+\sum_{i=1}^{k}\frac{b^i\ (t-ih)^i}{i!}e^{A_0(t-ih)},\ t\in J_k, k\ge 2 \quad (35)$$

The expressions (31), (32) and (35) complete the proof of the corollary.

3.2 Corollary 2

If $A_0=0$, then:

$$Y(t)=\begin{cases} I_n,\ t\in J_0;\\ I_n+\sum_{i=1}^{k}\frac{A_1^i(t-ih)^i}{i!},\ t\in J_k, k\ge 1\end{cases}$$

Proof

The proof follows by straight-forward successive integration, noting that $A_0=0\Rightarrow e^{A_0(.)}=I_n\Rightarrow A_1$ commutes with $e^{A_0(.)}=I_n$; So $Y(t)$ reduces to the result obtained by replacing b by A_1 and $e^{A_0(.)}$ by I_n in corollary 1. The desired result is precisely that stated in the conclusion of the corollary.

3.3 Corollary 3

If $n=1, A_0=a, A_1=b$, then:

$$Y(t)=\begin{cases} e^{at},\ t\in J_0;\\ e^{at}+\sum_{i=1}^{k}\frac{b^i\ (t-ih)^i}{i!}e^{a(t-ih)},\ t\in J_k, k\ge 1\end{cases}$$

Proof

Proof follows immediately by replacing A_0 by a and $\mathrm{diag}(b)$ by b in corollary 1.

3.4 Remarks: Please note in particular that this result is consistent with the theorem in Ukwu and Garba (2013c). However it must be pointed out that it was that theorem that motivated our theorem and hence corollary 3.

III. CONCLUSION

This article has completely determined the structure solution matrices which are indispensible for the determination of all solutions of single-delay autonomous differential and control systems. Moreover the ingenious combinations of summation notations, multiple product notations, multiple integrals and change of variable technique are unprecedented in the achievement of the desired proofs. The ideas exposed in this paper can be exploited to extend the results to double-delay and neutral systems.

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