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CONSTRUCTION OF OPTIMAL EXPRESSIONS FOR TRANSITION MATRICES OF A CLASS OF DOUBLE – DELAY SCALAR DIFFERENTIAL EQUATIONS

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ABSTRACT

This paper constructed optimal expressions for solution matrices of a class of double – delay autonomous linear differential equations on arbitrary intervals of length equal to the delay *h* for non –negative time periods. The proof was achieved using skilful combinations of summation notations, multinomial distribution, greatest integer functions, change of variables techniques, multiple integrals, as well as the method of steps to obtain these matrices on successive intervals of length equal to the delay *h*. This theorem globally extends the time scope of applications of these matrices to the solutions of initial function problems, rank conditions for controllability and cores of targets, constructions of controllability Grammians and admissible controls for transfers of points associated with controllability problems.

1. INTRODUCTION

There has been a flurry of activities in the qualitative approach to the controllability of functional differential control systems for the past fifty years among control theorists and applied mathematicians in general. This circumvents the severe difficulties associated with the search for and computations of solutions of such systems. Unfortunately computations of solutions cannot be wished away in the tracking of trajectories and many practical applications. Literature on state space approach to control studies is replete with variation of constants formulas, which incorporate the solution matrices of the free part of the systems. See Chukwu (1992), Gabsov and Kirillova (1976), Hale (1977), Manitius (1978), Tadmore (1984), and Ukwu (1987, 1992, 1996). Regrettably no author has made any attempt to obtain general expressions for such solution matrices or special cases of such matrices involving the double - delay h and $2h$. Effort is usually focused on the single – delay model and the approach has been to start from the interval $[0,h]$ and compute the solution matrices and solutions for given problem instances and then use the method of steps to extend these to the intervals $[kh,(k+1)h]$, for positive integral k, not exceeding 2, for the most part. Such approach is rather restrictive and doomed to failure in terms of structure for arbitrary k. In other words such approach fails to address the issue of the structure of solution matrices and solutions quite vital for real-world applications. The need to address such short-comings has become imperative; this is the major contribution of this paper, with limitations to scalar equations and wide-ranging implications for extensions to systems and holistic approach to controllability studies.

2. THEORETICAL ANALYSIS

We consider the class of double-delay differential equations

$$\dot{x}(t) = ax(t) + bx(t-h) + cx(t-2h),\ t \in \mathbf{R} \tag{1}$$

where a, b and c are arbitrary constants.

Let $Y(t)$ be a generic solution matrix of (1) for any $t \in \mathbf{R}$ and let $Y_{k-i}(t-ih)$ be a solution matrix of (1) on the interval

$$J_{k-i} = [(k-i)h, (k+1-i)h], k \in \{0,1,\cdots\}, i \in \{0,1,2\}, \tag{2}$$

where

$$Y(t) = \begin{cases} 1, \; t = 0, \\ 0, \; t < 0. \end{cases} \tag{3}$$

The unique solution matrix of (1) satisfying (3) is referred to as the transition matrix of (1).

By the definition of solution matrices,

$$\dot{Y}(t) = aY(t) + bY(t-h) + cY(t-2h) \; a.e, \; t \in \mathbf{R} \tag{4}$$

Hence

$$e^{-at}\left[\dot{Y}(t) - aY(t)\right] = \frac{d}{dt}\left[e^{-at}Y(t)\right] = bY(t-h) + cY(t-2h) \; a.e \tag{5}$$

The solution matrices will be obtained piece – wise on successive intervals of length h.

The objective of this paper is to formulate and prove a theorem on the general expression for $Y(t)$ on J_k, for $k \in \{0, 1, \cdots\}$, by appropriating the above expression for $Y(t)$, for the scalar equivalent: $n = 1, A_0 = a, A_1 = b$ and $A_2 = c$.

Let r_0, r_1, r_2 be nonnegative integers and let $P_{0(r_0),1(r_1),2(r_2)}$ denote the set of all permutations of $\underbrace{0,0,\ldots 0}_{r_0 \text{ times}} \; \underbrace{1,1,\ldots 1}_{r_1 \text{ times}} \; \underbrace{2,2,\ldots 2}_{r_2 \text{ times}}$: the permutations of the objects $0, 1,$ and 2 in which i appears r_i times, $i \in \{0,1,2\}$.

3. RESULTS AND ANALYSIS

3.1 Theorem: The Transition Matrix Formula for Autonomous, Double – Delay Linear Differential Equations (1)

Let $Y(t)$ be any solution matrix of (1) satisfying (3) and let J_k be as defined in (2). Then

$$Y(t) = \begin{cases} e^{at}, \; t \in J_0; & (6) \\ e^{at} + \sum_{i=1}^{k} b^i \frac{(t-ih)^i}{i!} e^{a(t-ih)} + \sum_{j=1}^{\left[\left[\frac{k}{2}\right]\right]} \sum_{i=0}^{k-2j} \frac{b^i c^j}{i!\,j!} \left(t - [i+2j]h\right)^{i+j} e^{a(t-[i+2j]h))}; \; t \in J_k, k \geq 1 & (7) \end{cases}$$

Remarks: Setting $a = a_0$, $b = a_1$ and $c = a_2$, and appealing to the multinomial distribution, the third summation components in (7) may be recast in the form

$$\sum_{j=1}^{\left[\left[\frac{k}{2}\right]\right]}\sum_{i=0}^{k-2j}\frac{b^i c^j}{i!\,j!}\left(t-[i+2j]h\right)^{i+j} e^{a(t-[i+2j]h))}$$

$$= \sum_{j=1}^{\left[\left[\frac{k}{2}\right]\right]}\sum_{i=0}^{k-2j}\sum_{(v_1,v_2,\cdots,v_{i+j})\in P_{1(i),2(j)}} a_{v_1}a_{v_2}\cdots a_{v_{i+j}}\frac{\left(t-[i+2j]h\right)^{i+j}}{(i+j)!}e^{a(t-[i+2j]h)} \qquad (8)$$

noting that the number of permutations of the integers 1 and 2 in which 1 and 2 appear i and j times respectively is simply

$\binom{i+j}{i}=\binom{i+j}{j}=\frac{(i+j)!}{i!\,j!}$. Hence $\frac{1}{(i+j)!}\frac{(i+j)!}{i!\,j!}=\frac{1}{i!\,j!}$, justifying (8) since a_1 and a_2

commute, being scalars. It is a herculean task, if not impossible to prove the theorem without using this equivalent form.

Note also that the formula above can be rewritten in the form

$$Y(t)=e^{at}+\sum_{i=1}^{k}b^i\frac{(t-ih)^i}{i!}e^{a(t-ih)}\operatorname{sgn}\left(\max\{k,0\}\right)$$

$$+\left[\sum_{j=1}^{\left[\left[\frac{k}{2}\right]\right]}\sum_{i=0}^{k-2j}\frac{b^i c^j}{i!\,j!}\left(t-[i+2j]h\right)^{i+j} e^{(t-[i+2j]h)}\right]\operatorname{sgn}\left(\max\{k-1,0\}\right), t\in J_k, k\geq 0 \qquad (9)$$

Furthermore, the sign restriction on k may be waived by rewriting the above formula in the form

$$Y(t)= e^{at}\operatorname{sgn}\left(\max\{k+1,0\}\right)+\sum_{i=1}^{k}b^i\frac{(t-ih)^i}{i!}e^{a(t-ih)}\operatorname{sgn}\left(\max\{k,0\}\right)$$

$$+\left[\sum_{j=1}^{\left[\left[\frac{k}{2}\right]\right]}\sum_{i=0}^{k-2j}\frac{b^i c^j}{i!\,j!}\left(t-[i+2j]h\right)^{i+j} e^{(t-[i+2j]h)}\right]\operatorname{sgn}\left(\max\{k-1,0\}\right), t\in J_k.$$

Proof

First, we prove that the theorem is true for $t\in J_k, k\in\{0,1,2\}$ Then we use induction to complete the proof.

From (1), $t\in J_0, Y(t-h)=Y(t-2h)=0\Rightarrow \dot{Y}(t)=a_0Y(t)$ a.e. $\Rightarrow Y(t)\equiv Y_1(t)=e^{a_0t}C$; $Y(0)=I_n\Rightarrow C=I_n\Rightarrow Y(t)=e^{a_0t}$. From (6), $t\in J_0\Rightarrow Y(t)=e^{a_0t}$. So the theorem is true for $t\in J_0$. Observe that $\lim_{t\to 0^-}Y(t)=0$ and $\lim_{t\to 0^+}Y(t)=1\Rightarrow Y(t)$ is discontinuous at 0.

Consider the t-interval $(h,2h)\subset J_1$. Then $t-h\in(0,h), t-2h\in(-h,0)\Rightarrow Y(t-2h)=0$

$$\Rightarrow \dot{Y}(t)=a_0Y(t)+a_1e^{a_0(t-h)}\Rightarrow \frac{d}{dt}\left[e^{-a_0}Y(t)\right]=e^{-a_0t}\left[\dot{Y}(t)-a_0Y(t)\right]=e^{-a_0t}Y(t)=a_1e^{-a_0t}e^{a_0(t-h)}$$

$$=a_1e^{-a_0h}\Rightarrow e^{-a_0t}Y(t)=e^{-a_0h}Y(h)+\int_h^t a_1e^{-a_0h}ds=1+a_1(t-h)e^{-a_0h}\Rightarrow Y(t)=e^{a_0t}+a_1(t-h)e^{a_0(t-h)},$$

for $t\in J_1$.

From (8), $t \in J_1 \Rightarrow$

$$Y(t) = e^{a_0 t} + a_1(t-h)e^{a_0(t-h)} + \sum_{j=1}^{0}\sum_{i=0}^{1-2j} \sum_{(v_1,v_2,\cdots,v_{i+j})\in P_{1(i),2(j)}} a_{v_1}a_{v_2}\cdots a_{v_{i+j}} \frac{\left(t-[i+2j]h\right)^{i+j}}{(i+j)!}$$

$= e^{a_0 t} + a_1(t-h)e^{a_0(t-h)}$, since the double summation is infeasible.

Therefore the theorem is valid for $t \in J_1$.

Using (1) and the continuity of $Y(t)$ for $t > 0$, $\lim_{t\to h^+}\dot{Y}(t) = a_0 e^{A_0 h} + a_1 = \lim_{t\to h^-}\dot{Y}(t) = \dot{Y}(h) = 0$ $\Rightarrow Y(t)$ is differentiable at $t = h$.

Consider the interval $(2h,3h) \subset J_2$; $t \in (2h,3h), s_2 \in (2h,t]$. Then

$$t-h \in (h,2h), t-2h \in (0,\ h) \Rightarrow s_2 - h \in (h,2h), s_2 - 2h \in (0,\ h)$$

$$\Rightarrow Y(t-2h) = e^{a_0(t-2h)},\ \ Y(t-h) = e^{a_0(t-h)} + a_1(t-2h)e^{a_0(t-2h)}$$

$$\Rightarrow\ \dot{Y}(t) - a_0 Y(t) = a_1 e^{a_0(t-h)} + a_1^2(t-2h)e^{a_2(t-2h)} + a_2 e^{a_0(t-2h)},\ \text{on } (2h,3h).$$

$$\int_{2h}^{t}\left(\frac{d}{ds_2}\left[e^{-a_0 s_2}Y(s_2)\right]\right)ds_2 = \int_{2h}^{t} e^{-a_0 s_2}\left(\dot{Y}(s_2) - a_0 Y(s_2)\right)ds_2$$

$$\Rightarrow Y(t) = e^{a_0(t-2h)}Y(2h) + a_1\int_{2h}^{t} e^{a_0(t-s_2)}Y(s_2-h)ds_2 + a_2\int_{2h}^{t} e^{A_0(t-s_2)}Y(s_2-2h)ds_2 .$$

Therefore

$$Y(t) = e^{a_0(t-2h)}\left[e^{2a_0 h} + a_1 h e^{a_0 h}\right] + a_1\int_{2h}^{t} e^{a_0(t-s_2)}\left[e^{a_0(s_2-h)} + a_1(s_2-2h)e^{a_0(s_2-2h)}\right]ds_2$$
$$+ a_2\int_{2h}^{t} e^{a_0(t-s_2)}e^{a_0(s_2-2h)}ds_2$$

$$\Rightarrow Y(t) = e^{a_0(t-2h)}\left[e^{2a_0 h} + a_1 h e^{a_0 h}\right] + a_1\int_{2h}^{t} e^{a_0(t-s_2)}\left[e^{a_0(s_2-h)} + a_1(s_2-2h)e^{a_0(s_2-2h)}\right]ds_2$$
$$+ a_2\int_{2h}^{t} e^{a_0(t-s_2)}e^{a_0(s_2-2h)}ds_2$$

$$\Rightarrow Y(t) = e^{a_0 t} + a_1 h e^{a_0(t-h)} + a_1(t-2h)e^{a_0(t-h)} + a_1^2\int_{2h}^{t}(s_2-2h)e^{a_0(t-2h)}ds_2\ + a_2(t-2h)e^{a_0(t-2h)}$$

$$\Rightarrow Y(t) = e^{a_0 t} + a_1(t-h)e^{a_0(t-h)} + a_1^2\frac{(t-2h)^2}{2!}e^{a_0(t-2h)}\ + a_2(t-2h)e^{a_0(t-2h)}$$

$$\Rightarrow Y(t) = e^{a_0 t} + \sum_{i=1}^{2}\frac{a_1^i(t-ih)^i}{i!}e^{a_0(t-ih)}\ + a_2(t-2h)e^{a_0(t-2h)}$$

From (8), $t \in J_2 \Rightarrow k = 2 \Rightarrow$

$$Y(t) = e^{a_0 t} + \sum_{i=1}^{2} a_1^i \frac{(t-ih)^i}{i!} e^{a_0(t-ih)} + \sum_{j=1}^{1}\sum_{i=0}^{0} \sum_{(v_1,v_2,\cdots,v_{i+j})\in P_{1(i),2(j)}} a_{v_1} a_{v_2} \cdots a_{v_{i+j}} \frac{(t-[i+2j]h)^{i+j}}{(i+j)!} e^{a_0(t-2h)}$$

$$\Rightarrow Y(t) = e^{a_0 t} + \sum_{i=1}^{2} a_1^i \frac{(t-ih)^i}{i!} e^{a_0(t-ih)} + a_2(t-2h)e^{a_0(t-2h)},\ t \in J_2.$$

Therefore the theorem is also valid on J_2; hence, the theorem has been be verified for $k \in \{0,1,2\}$. Assume that the theorem is valid for $t \in J_p, 3 \le p \le k$, for some integers p and k. Then $t, s_{k+1} \in J_{k+1} \Rightarrow [k+1]h \in J_k$, $s_{k+1} - h \in J_k$ and $s_{k+1} - 2h \in J_{k-1}$. Hence

$$Y(t) = Y([k+1]h)e^{a_0(t-[k+1]h)} + \int_{[k+1]h}^{t} e^{a_0(t-s_{k+1})} a_1 Y(s_{k+1}-h)ds_{k+1} + \int_{[k+1]h}^{t} e^{a_0(t-s_{k+1})} a_2 Y(s_{k+1}-2h)ds_{k+1} \quad (10)$$

$$= e^{a_0 t} + \sum_{i=1}^{k} \frac{a_1^i([k+1-i]h)^i}{i!} e^{a_0(t-ih)} \quad (11)$$

$$+ \sum_{j=1}^{\left[\left[\frac{k}{2}\right]\right]} \sum_{i=0}^{k-2j} \sum_{(v_1,v_2,\cdots,v_{i+j})\in P_{1(i),2(j)}} a_{v_1} a_{v_2} \cdots a_{v_{i+j}} \frac{([k+1-i-2j]h)^{i+j}}{(i+j)!} e^{a_0(t-[i+2j]h)} \quad (12)$$

$$+ \int_{[k+1]h}^{t} a_1 \left[e^{a_0(t-h)} + \sum_{i=1}^{k} \frac{a_1^i(s_{k+1}-[i+1]h)^i}{i!} e^{a_0(t-[i+1]h)} \right] ds_{k+1} \quad (13)$$

$$+ \int_{[k+1]h}^{t} a_1 \left[\sum_{j=1}^{\left[\left[\frac{k}{2}\right]\right]} \sum_{i=0}^{k-2j} \sum_{(v_1,v_2,\cdots,v_{i+j})\in P_{1(i),2(j)}} a_{v_1} a_{v_2} \cdots a_{v_{i+j}} \frac{(s_{k+1}-[1+i+2j]h)^{i+j}}{(i+j)!} e^{a_0(t-[i+1+2j]h))} \right] \quad (14)$$

$$+ \int_{[k+1]h}^{t} a_2 \left[e^{a_0(t-h)} + \sum_{i=1}^{k-1} \frac{a_1^i(s_{k+1}-[i+2]h)^i}{i!} e^{a_0(t-[i+2]h)} \right] ds_{k+1} \quad (15)$$

$$+ \int_{[k+1]h}^{t} a_2 \left[\sum_{j=1}^{\left[\left[\frac{k-1}{2}\right]\right]} \sum_{i=0}^{k-1-2j} \sum_{(v_1,v_2,\cdots,v_{i+j})\in P_{1(i),2(j)}} a_{v_1} a_{v_2} \cdots a_{v_{i+j}} \frac{(s_{k+1}-[i+2+2j]h)^{i+j}}{(i+j)!} e^{a_0(t-[i+2+2j]h))} \right] \quad (16)$$

The expression (13) yields

$$a_1(t-[k+1]h)e^{a(t-h)} + \sum_{i=2}^{k+1} \frac{a_1^i(t-ih)^i}{i!} e^{a_0(t-ih)} - \sum_{i=2}^{k+1} \frac{a_1^i([k+1-i])^i}{i!} e^{a_0(t-ih)} \quad (17)$$

The expression (14) yields

$$a_1 \left[\sum_{j=1}^{\left[\left[\frac{k}{2}\right]\right]} \sum_{i=0}^{k+1-2j} \sum_{(v_1,v_2,\cdots,v_{i+j})\in P_{1(i-1),2(j)}} a_{v_1} a_{v_2} \cdots a_{v_{i+j}} \frac{(s_{k+1}-[i+2j]h)^{i+j}}{(i+j)!} e^{a_0(t-[i+2j]h))} \right] \quad (18)$$

$$- a_1 \sum_{j=1}^{\left[\left[\frac{k}{2}\right]\right]} \sum_{i=0}^{k+1-2j} \sum_{(v_1,v_2,\cdots,v_{i+j})\in P_{1(i-1),\,2(j)}} a_{v_1} a_{v_2} \cdots a_{v_{i-1+j}} \frac{([k+1-i-2j]h)^{i+j}}{(i+j)!} e^{a_0(t-[i+2j]h))} \quad (19)$$

since the summations with $i = 0$ are infeasible and so may be equated to zero, yielding

$$\sum_{j=1}^{\left[\left[\frac{k}{2}\right]\right]} \sum_{i=0}^{k+1-2j} \sum_{(v_1,v_2,\cdots,v_{i+j})\in P_{1(i),\,2(j)}} a_{v_1} a_{v_2} \cdots a_{v_{i+j}} \frac{\left(t-[i+2j]h\right)^{i+j}}{(i+j)!}, \tag{20}$$

(with a leading $a_1 = b$)

$$-\sum_{j=1}^{\left[\left[\frac{k}{2}\right]\right]} \sum_{i=0}^{k+1-2j} \sum_{(v_1,v_2,\cdots,v_{i+j})\in P_{1(i),\,2(j)}} a_{v_1} a_{v_2} \cdots a_{v_{i+j}} \frac{\left([k+1-i-2j]h\right)^{i+j}}{(i+j)!}, \tag{21}$$

(with a leading $a_1 = b$)

The expression (15) yields

$$a_2\left(t-[k+1]h\right)e^{a_0(t-h)} + \sum_{i=1}^{k-1} \frac{a_2 a_1^i \left(t-[i+2]h\right)^{i+1}}{(i+1)!} e^{a_0(t-[i+2]h)} - \sum_{i=1}^{k-1} \frac{a_2 a_1^i \left([k-1-i]h\right)^{i+1}}{(i+1)!} e^{a_0(t-[i+2]h)} \tag{22}$$

$$= a_2\left([1-k]h\right)e^{a_0(t-h)} + \sum_{i=0}^{k+1-2(1)} \frac{a_2 a_1^i \left(t-[i+2]h\right)^{i+1}}{(i+1)!} e^{a_0(t-[i+2]h)} - \sum_{i=1}^{k-1} \frac{a_2 a_1^i \left([k-1-i]h\right)^{i+1}}{(i+1)!} e^{a_0(t-[i+2]h)}$$

The expression (16) yields

$$\sum_{j=2}^{\left[\left[\frac{k+1}{2}\right]\right]} \sum_{i=0}^{k+1-2j} \sum_{(v_1,v_2,\cdots,v_{i+j})\in P_{1(i),\,2(j)}} a_{v_1} a_{v_2} \cdots a_{v_{i+j}} \frac{\left(t-[i+2j]h\right)^{i+j}}{(i+j)!} e^{a_0(t-[i+2j])}, \tag{23}$$

(with a leading $a_2 = c$)

$$-\sum_{j=2}^{\left[\left[\frac{k+1}{2}\right]\right]} \sum_{i=0}^{k+1-2j} \sum_{(v_1,v_2,\cdots,v_{i+j})\in P_{1(i),\,2(j)}} a_{v_1} a_{v_2} \cdots a_{v_{i+j}} \frac{\left([k+1-i-2j]h\right)^{i+j}}{(i+j)!} e^{a(t-[i+2j])}, \tag{24}$$

(with a leading a_2)

$$= \sum_{j=1}^{\left[\left[\frac{k+1}{2}\right]\right]} \sum_{i=0}^{k+1-2j} \sum_{(v_1,v_2,\cdots,v_{i+j})\in P_{1(i),\,2(j)}} a_{v_1} a_{v_2} \cdots a_{v_{i+j}} \frac{\left(t-[i+2j]h\right)^{i+j}}{(i+j)!} e^{a(t-[i+2j])}, \tag{25}$$

(with a leading a_2)

$$- \sum_{i=0}^{k-1} a_2 a_1^i \frac{\left(t-[i+2]h\right)^{i+1}}{(i+1)!} e^{a_0(t-[i+2]h)} \tag{26}$$

$$-\sum_{j=1}^{\left[\left[\frac{k+1}{2}\right]\right]} \sum_{i=0}^{k+1-2j} \sum_{(v_1,v_2,\cdots,v_{i+j})\in P_{1(i),\,2(j)}} a_{v_1} a_{v_2} \cdots a_{v_{i+j}} \frac{\left([k+1-i-2j]h\right)^{i+j}}{(i+j)!} e^{a(t-[i+2j])}, \tag{27}$$

(with a leading a_2)

$$+ \sum_{i=0}^{k-1} a_2 a_1^i \frac{\left([k-1-i]h\right)^{i+1}}{(i+1)!} e^{a_0(t-[i+2j])} \tag{28}$$

Therefore

$$Y(t) = (11)+(12)+(17)+(20)+(21)+(22)+(25)+(26)+(27)+(28) = (29)+(30)+\cdots+(38)$$

$$= e^{a_0 t} + \sum_{i=1}^{k} \frac{a_1^i\left([k+1-i]h\right)^i}{i!} e^{a_0(t-ih)} \tag{29}$$

$$+ \sum_{j=1}^{\left[\left[\frac{k+1}{2}\right]\right]} \sum_{i=0}^{k+1-2j} \sum_{(v_1,v_2,\cdots,v_{i+j})\in P_{1(i),\,2(j)}} a_{v_1}a_{v_2}\cdots a_{v_{i+j}} \frac{\left([k+1-i-2j]h\right)^{i+j}}{(i+j)!} e^{a(t-[i+2j]h)} \tag{30}$$

$$+ a_1(t-[k+1]h)e^{a_0(t-h)} + \sum_{i=2}^{k+1} \frac{a_1^i\left(t-ih\right)^i}{i!} e^{a_0(t-ih)} - \sum_{i=2}^{k+1} \frac{a_1^i\left([k+1-i]h\right)^i}{i!} e^{a_0(t-ih)} \tag{31}$$

$$+ \sum_{j=1}^{\left[\left[\frac{k+1}{2}\right]\right]} \sum_{i=0}^{k+1-2j} \sum_{(v_1,v_2,\cdots,v_{i+j})\in P_{1(i),\,2(j)}} a_{v_1}a_{v_2}\cdots a_{v_{i+j}} \frac{\left(t-[i+2j]h\right)^{i+j}}{(i+j)!} e^{a(t-[i+2j]h)}, \tag{32}$$

$$\left(\text{with a leading } a_1, \text{noting that } k \text{ even} \Rightarrow \left[\left[\frac{k}{2}\right]\right] = \left[\left[\frac{k}{2}\right]\right];\ k \text{ odd } \Rightarrow \left[\left[\frac{k}{2}\right]\right] = \left[\left[\frac{k+1}{2}\right]\right] - 1 \text{ and}\right)$$

$$\left(k \text{ odd},\ j = \left[\left[\frac{k+1}{2}\right]\right] \Rightarrow k+1-2j=0 \Rightarrow \sum_{i=0}^{k+1-2j}(.) = 0, \text{being infeasible. So } \sum_{j=1}^{\left[\left[\frac{k+1}{2}\right]\right]}(.) \text{is appropriate.}\right)$$

$$- \sum_{j=1}^{\left[\left[\frac{k+1}{2}\right]\right]} \sum_{i=0}^{k+1-2j} \sum_{(v_1,v_2,\cdots,v_{i+j})\in P_{1(i),\,2(j)}} a_{v_1}a_{v_2}\cdots a_{v_{i+j}} \frac{\left([k+1-i-2j]h\right)^{i+j}}{(i+j)!} e^{a(t-[i+2j]h)}, \tag{33}$$

(with a leading a_1)

$$+a_2\left(t-[k+1]h\right)^{a_0(t-h)} + \sum_{i=1}^{k-1} \frac{a_2 a_1^i\left(t-[i+2]h\right)^{i+1}}{(i+1)!} e^{a_0(t-[i+2]h)} - \sum_{i=1}^{k-1} \frac{a_2 a_1^i\left([k-1-i]h\right)^{i+1}}{(i+1)!} e^{a_0(t-[i+2]h)} \tag{34}$$

$$+ \sum_{j=1}^{\left[\left[\frac{k+1}{2}\right]\right]} \sum_{i=0}^{k+1-2j} \sum_{(v_1,v_2,\cdots,v_{i+j})\in P_{1(i),\,2(j)}} a_{v_1}a_{v_2}\cdots a_{v_{i+j}} \frac{\left(t-[i+2j]h\right)^{i+j}}{(i+j)!} e^{a(t-[i+2j]h)}, \tag{35}$$

(with a leading a_2)

$$- \sum_{i=0}^{k-1} a_2 a_1^i \frac{\left(t-[i+2]h\right)^{i+1}}{(i+1)!} e^{a_0(t-[i+2]h)} \tag{36}$$

$$- \sum_{j=1}^{\left[\left[\frac{k+1}{2}\right]\right]} \sum_{i=0}^{k+1-2j} \sum_{(v_1,v_2,\cdots,v_{i+j})\in P_{1(i),\,2(j)}} a_{v_1}a_{v_2}\cdots a_{v_{i+j}} \frac{\left([k+1-i-2j]h\right)^{i+j}}{(i+j)!} e^{a_0(t-[i+2j]h)}, \tag{37}$$

(with a leading c)

$$+ \sum_{i=0}^{k-1} a_2 a_1^i \frac{\left([k-1-i]h\right)^{i+1}}{(i+1)!} \tag{38}$$

(29) + (31) yields

$$e^{a_0 t} + \sum_{i=1}^{k+1} \frac{a_1^i \left(t-ih\right)^i}{i!} e^{a_0(t-ih)} \qquad (39)$$

(32) + (35) yields

$$\sum_{j=1}^{\left[\left[\frac{k+1}{2}\right]\right]} \sum_{i=0}^{k+1-2j} \sum_{(v_1,v_2,\cdots,v_{i+j})\in P_{1(i),\,2(j)}} a_{v_1} a_{v_2} \cdots a_{v_{i+j}} \frac{\left(t-[i+2j]h\right)^{i+j}}{(i+j)!} e^{a_0(t-[i+2j])} \qquad (40)$$

Expressions (30) + (33) + (37) yield zero; the expressions cancel out.

Expressions (34) + (34) + (38) yield zero; the expressions cancel out.

Therefore, on J_{k+1}, $Y(t)$ reduces to $Y(t) =$ expression (39) + expression (40). Thus

$$t \in J_{k+1} \Rightarrow Y(t) = e^{a_0 t} + \sum_{i=1}^{k+1} \frac{a_1^i \left(t-ih\right)^i}{i!} e^{a_0(t-ih)}$$
$$+ \sum_{j=1}^{\left[\left[\frac{k+1}{2}\right]\right]} \sum_{i=0}^{k+1-2j} \sum_{(v_1,v_2,\cdots,v_{i+j})\in P_{1(i),\,2(j)}} a_{v_1} a_{v_2} \cdots a_{v_{i+j}} \frac{\left(t-[i+2j]h\right)^{i+j}}{(i+j)!} e^{a_0(t-[i+2j]h)} \qquad (41)$$

$$\Rightarrow Y(t) = e^{a_0 t} + \sum_{i=1}^{k+1} a_1^i \frac{\left(t-ih\right)^i}{i!} e^{a_0(t-ih)} + \sum_{j=1}^{\left[\left[\frac{k+1}{2}\right]\right]} \sum_{i=0}^{k+1-2j} \frac{a_1^i a_2^j}{i!\,j!} \left(t-[i+2j]h\right)^{i+j} e^{a_0\left(t-[i+2j]h\right))} \qquad (42)$$

This completes the proof of the theorem.

CONCLUSION

This article addressed the issue of the structure of the transition matrices of (1) on arbitrary intervals of length equal to the delay $h,$ obviating the need to start from the interval $[0,h]$ in order to compute the transition matrices and solutions for given problem instances and then use successively the method of steps to extend these to the intervals $[kh,(k+1)h],$ for positive integral $k.$ By applying any of the alternative formulas on the interval $[kh,(k+1)h],\ k \in \{0,1,\cdots\},$ the solutions of initial function problems associated with (1) can be more readily obtained. Furthermore controllability Grammians and admissible controls for transfers of points associated with controllability problems can be easily constructed. $[kh,(k+1)h], k \in \{0,1,\cdots\},$

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