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Zero-based batch starting age algorithm for global optimal strategies and returns for a class of Stationary equipment replacement problems with age transition perspectives

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Abstract: This research article conceptualized, formulated and designing an Excel automated solution-based algorithm for the optimal policy prescriptions and the corresponding returns for all batches of feasible starting ages for a class of equipment replacement problems with stationary pertinent data. The tasks were accomplished by the exploitation of the structure of the states given as functions of the decision periods, and the use of starting age index zero, in age-transition dynamic programming recursions. The investigation revealed that if m is a fixed replacement age in a base problem with horizon length n, and a single starting age $t_1 = 0,\ n_2 \geq n$ may be selected such that the optimal solutions and corresponding rewards for the n_2 - stage problem from stage $1+n_2-n$ to stage n_2 coincide with those of the revised base problem with any batch of feasible nonzero starting ages in stage $1+n_2-n$ of the n_2- stage problem. By an appeal to the structure of the states at each stage and the deployment of the preliminary starting age $t_1 = 0$ master stroke in the n_2- stage problem, the optimal policy prescriptions and rewards for the base problem for the full set $\{1, 2, \cdots, m\}$ of feasible starting ages coincide with those of the n_2- stage problem from stage $1+n_2-n$ to stage n_2, resulting in m different problems being solved at once. The paper concludes that, if $n < n_2$, such that $n_2 - n \geq m$, then $D_j^*(s_j),\ f_j(s_j)$ are stage j optimal decisions and rewards from the template with horizon length n_2, for $j \in \{n_2+1-n, \cdots, n_2\}$ if and only if $D^*_{j+n-n_2}(s_j)$ and $f_{j+n-n_2}(s_j)$ are the corresponding optimal decisions and rewards in stage $j+n-n_2$ for the template with the horizon length n and revised set $\{1, 2, \cdots, m\}$, of starting ages. Moreover, the optimal decisions and corresponding rewards for the base problem are immediate from the choice $n_2 = n$.

Keywords: Dynamic programming recursions, Age transition diagrams, Batch Automation of optimality results, Decision symbols, Decision period, Equipment Replacement Problems, One fell swoop, Pertinent Data, Sensitivity Analyses.

1. INTRODUCTION

The Equipment Replacement Problem is an area of acute research need and of considerable and diverse research interests. Every equipment undergoes wear and tear and is subject to obsolescence with increasing time, resulting in deterioration and compromise in its performance characteristics. The need for restorative remediation becomes imperative. However, the associated rising operating and maintenance costs, as well as the decreasing salvage values and revenue generation capacity necessitate the replacement of the equipment at the appropriate time at some trade-in value. Therefore optimal replacement decisions must be made to optimize the returns.

Consider the problem of researching an optimal Equipment Replacement policy over an n - period planning horizon. At the start of each year a decision is made whether to keep the equipment in service an extra year or to replace it with a new one at some salvage value. Fan et al. [1] remarked that the primary function of equipment managers is to replace the right equipment at the time and at the lowest cost. They went on to discuss among other things, the opportunities and challenges associated with equipment replacement decision making. Fallahnezhad et al. [2], presented an optimal decision rule for minimizing total cost in designing a sampling plan for machine replacement problems using dynamic programming and Bayesian inferential approaches. The cost of replacing the machine and the cost of produced defectives were factored into the model, and the concept of control threshold policy was applied in the decision rule as follows: If the probability of producing a defective exceeded the control threshold, then the machine was replaced, otherwise the production system would be deemed to be in a state of statistical control and production would go on uninterrupted. Finally, the paper presented a numerical example as well as performed sensitivity analysis to illustrate the application of their result. Zvipore et al. [3] investigated the application of stationary equipment replacement dynamic programming model in conveyor belt replacement using a Gold mining company in Zimbabwe as a case study. Their findings revealed that conveyor belts should be replied in the mining system on a yearly basis and concluded that equipment replacement policy for conveyor belts is a necessity in a mining system, so as to achieve optimum contribution to the economic value that a mining system might accrue within a period of time. Fan et al. [4] formulated a stochastic dynamic programming-based optimization model for the equipment replacement problem that could explicitly account for the uncertainty in vehicle utilization.

As remarked by [5], "the determination of the feasible values for the age of the machine at each stage is somewhat tricky". The latter went on to obtain the optimal replacement ages using network diagrammatic approach, with machine ages on the vertical axis and decision years on the horizontal axis. In an alternative time perspective approach, [6] initiated the determination process for the optimal replacement time with network diagrams consisting of upper half-circles on the horizontal axis, initiating from each feasible time of the planning horizon and terminating at feasible times, with the length of successive transition times at most, the maximum operational age of the equipment. Sequel to this, [6] formulated dynamic recursions as functions of the decision times, the corresponding feasible transition times, the problem data and the cash-flow profile. Unfortunately network diagrams are unwieldy, cumbersome and prone to errors, especially for large problem instances; consequently the integrity of the desired optimal policies may be compromised. Verma [7] and [8] used the average annual cost criteria to determine alternative optimal policies and the corresponding optimal rewards in a non-dynamic programming fashion. Gress et al. [9] modeled the equipment replacement problem using a Markov decision process and a reward function that can be more helpful in processing industries. Unfortunately, the key issues of large-scale implementation and sensitivity analyses were not discussed by the afore-mentioned authors.

A new impetus was provided for sensitivity analyses and implementation paradigm shift by [10], with respect to optimal solutions to machine replacement problems. Ukwu [10] pioneered the development of computational formulas for the feasible states corresponding to each decision year in a certain class of equipment Replacement problems, thereby eliminating the drudgery and errors associated with the drawing of network diagrams for such determination. Ukwu [10] went further to design prototypical solution templates for optimal solutions to such problems, complete with an exposition on the interface and solution process. Ukwu [11] extended the formulations and results in [10] to a class of machine replacement problems, with pertinent data given as functions of machine ages and the decision periods of the planning horizon. By restructuring the data in three – dimensional formats [11] appropriated key features of the template in [10] for the extended template. Finally [11] solved four illustrative examples of the same flavour that demonstrated the efficiency, power and utility of the solution template prototype. In [11], it was pointed out that the template could be deployed to solve each equipment replacement problem in less than 10 percent of the time required for the manual generation of the alternate optima. However a major draw-back of the templates in [10, 11] is that for any problem instance, the inputs of the states and stage numbering are manually generated. Moreover, the templates require row updating of the formulas for the optimal criterion function values for problems of larger horizon lengths. Evidently this functionality needs to be improved upon for more speedy solution implementations, especially for practical problems of long planning horizons. Ukwu [12] used the state concept to obtain the structure of the sets of feasible replacement times corresponding to various decision times, in equipment replacement problems, thereby obviating the need for network diagrams for such determination. It went further to undertake novel formulations of the equipment replacement problems, incorporating cardinality analyses on the feasible transition states for each feasible time.

Furthermore, the article designed solution implementation templates for the corresponding dynamic programming recursions. These templates circumvent the inherent tedious and prohibitive manual computations associated with dynamic programming formulations and may be optimally appropriated for sensitivity analyses on such models in just a matter of minutes. Ukwu [13] examined the effects of different planning horizons, with equipment replacement age fixed, in the Excel automated solutions in [12], to a class equipment replacement problems with stationary pertinent data. The investigation revealed that if the replacement age is fixed, and n_1 and n_2 are any two horizon lengths with $n_1 < n_2$, and p^*_j, $g(j)$ are stage j optimal decision and reward from the template with horizon length n_2, for $j \in \{n_2+1-n_1, \cdots, n_2\}$, then $p^*_j + n_1 - n_2$ *and* $\mathrm{g}\left(j+n_1-n_2\right)$ are the corresponding optimal decision and reward in stage $j+n_1-n_2$ for the template with the horizon length n_1. Moreover the corresponding optimal rewards are equal. The results were achieved by the use of the structure of feasible replacement time sets and appropriate dynamic programming recursions. Ukwu [14] investigated the cases $t_1 \in \{1, 2, \cdots, m\}$, where m is the mandatory equipment replacement age and t_1 is the starting age of the equipment. In Ukwu [14] the cases $t_1 \ge 2$ required only trivial repositioning of the last automated state $i-1+t_1, t_1-1$ places to the right, with the cell values in-between deleted in each of stages $m+1+t_1, m+t_1, \ldots, 2, 1$ of the process, where $i \in \{1, 2, \cdots, n\}$ is the decision year and n is the length of the planning horizon. Ukwu [15] remedied the drawbacks in [10, 11] by providing alternative layout and solution templates to those in [10], with full automation of all computations for the case $t_1 = 0$. Ukwu [15] also gave an exposition on the solution template incorporating the outputs for the given problem and general problems in that class, and the solution outputs for the problem were shown to be consistent with the general exposition.

The major contributions of this article are as follows: It uses the starting age of zero in [8] to eliminate the manual intervention in the repositioning of the states in [14] and solves simultaneously, in a single action, any instance of the equipment replacement problem for the entire set $\{t_1\} = \{1, 2, \cdots, m\}$ of feasible nonzero starting ages. It also solves the n-stage problem with starting age 0. These are, indeed, trail-blazing scientific achievements, with far reaching implications for holistic and batch optimal solution implementations for large-scale multiple starting age problems.

2. MATERIAL AND METHODS

In this section, the problem data, working definitions, elements of the DP model and the dynamic programming (DP) recursions are laid out as follows:

Equipment Starting age $= t_1$

Equipment Replacement age $= m$

S_i = The set of feasible equipment ages (states) in decision period i (say year i); $i \in \{1, 2, \ldots, n\}$

$r(t)$ = annual revenue from a $t-$ year old equipment

$c(t)$ = annual operating cost of a $t-$ year old equipment

$s(t)$ = salvage value of a $t-$ year old equipment; $t = 0, 1, \ldots, m$

I = fixed cost of acquiring a new equipment in any year

The elements of the DP are the following:

1. Stage i, represented by year $i, i \in \{1, 2, \ldots, n\}$
2. The alternatives at stage (year) i. These call for keeping or replacing the equipment at the beginning of year i
3. The state at stage (year) i, represented by the age of the equipment at the beginning of year i.

Let $f_i(t)$ be the maximum net income for years $i, i+1, \ldots, n-1, n$ given that the equipment is t years old at the beginning of year i.

Note: The definition of $f_i(t)$ starting from year i to year n implies that backward recursion will be used. Forward recursion would start from year 1 to year i.

The template will appropriate the following theorem formulated in [5] and exploited in [10, 11], using backward recursive procedure.

2.1 Theorem 1: Dynamic Programming Recursions for Optimal Policy and Rewards [5]

$$f_i(t) = \max \begin{cases} r(t) - c(t) + f_{i+1}(t+1); \text{ IF KEEP} \\ r(0) + s(t) - I - c(0) + f_{i+1}(1); \text{ REPLACE} \end{cases}$$

$$f_{n+1}(x) = s(x), \quad i = 0, 1, \ldots, n-1, \ x = \text{age of machine at the start of period } n+1$$

Proof (Ukwu [10])

Decision: KEEP

$r_i(t) - c_i(t) =$ net revenue (income) from a t-year old machine during the decision year i. Then the equipment age advances to $t+1$ years and hence $f_{i+1}(t+1) =$ maximum income for years $i+1, \ldots, n$ given that the equipment is $t+1$ years old at the start of year $i+1$.

Decision:

$r_i(0) = 1-$ year revenue from a new equipment (age 0) during the decision year i.

$c_i(0)$ = cost of operating a new equipment for 1 year (from the start of year i to the end of year i)

REPLACE I_i = cost of a new equipment during the decision year i.

$s_i(t)$ = salvage cost for a $t-$ year old equipment during the decision year i.

$$\text{Net income} = \underbrace{r_i(0) - c_i(0)}_{\substack{\text{net income for operating} \\ \text{the new equipment from the} \\ \text{beginning of year } i \text{ to the} \\ \text{end of year } i\text{. The age of the} \\ \text{equipment then advances to 1 year}}} + s_i(t) - I_i + \underbrace{f_{i+1}(1)}_{\substack{\text{max net income for} \\ \text{years } i+1,\ldots,n \text{ given} \\ \text{that the machine is 1-year} \\ \text{old at the start of year } i+1}}$$

$f_{n+1}(x) = s_{n+1}(x)$ or $f_{n+1}(.) = s_{n+1}(.) \Rightarrow$ sell off the equipment at the end of the planning horizon at price $s_{n+1}(.)$, regardless of its age, with no further income realized from the beginning of year $n+1$, since the planning horizon length is n years. Therefore the recursive equation is correct. This completes the proof.

2.2 Pertinent Remarks on the DP Recursions(Ukwu [10])

For $i \in \{1, 2, \cdots, n\}$, $f_i(t)$ may be identified as $f_i(t) = \max_{\{K,R\}} \left\{ f_i^K(t), f_i^R(t) \right\}$, where

$f_i^K(t) = r_i(t) - c_i(t) + f_{i+1}(t+1)$ and $f_i^R(t) = r_i(0) + s_i(t) - I_i - c_i(0) + f_{i+1}(1)$

For $i \in \{1, 2, \cdots, n\}$ and $t \in S_i$ the optimal decision may be identified as $D_i(t)$, where

$$D_i(t) = \underset{\{K,R\}}{\operatorname{argmax}}\, g_i(t, K, R);\; g_i(t, K, R) = \begin{cases} f_i^K(t), & \text{if Decision is KEEP} \\ f_i^R(t), & \text{if Decision is REPLACE} \end{cases}$$

Define

$$x_i = \begin{cases} 1, & \text{if decision is REPLACE in stage } i \text{ (start of decision year } i) \\ 0, & \text{if decision is KEEP in stage } i \text{ (start of decision year } i) \end{cases}$$

Then

$$g_i(t, K, R) = f_i(t) = (1 - x_i) f_i^K(t) + x_i f_i^R(t),\; i \in \{1, 2, \cdots, n\}$$

If the revenue profile is not given, then $r_i(t)$ may be set identically equal to zero, in which case

$-f_i(t) =$ minimum cost associated with operating the equipment from the start of decision year i to the end of decision year n.

If the variable cost profile is not given then $c_i(t)$ may be set identically equal to zero, in which case

$f_i(t) =$ maximum net revenue (maximum profit) from the start of decision year i to the end of decision year n.

If the cost profile is not given then $c_i(t)$ and I_i may be set identically equal to zero, in which case

$f_i(t) =$ maximum accrueable revenue from the start of decision year i to the end of decision year n.

3. RESULTS AND DISCUSSION

3.1 Theorem on Analytic Determination of the Set of Feasible Ages at Each Stage. Ukwu [10]

Let S_i denote the set of feasible equipment ages at the start of the decision year i. Let t_1 denote the age of the machine at the start of the decision year i, that is, $S_1 = \{t_1\}$. Then for $i \in \{1, 2, \ldots, n\}$,

$$S_i = \begin{cases} \left\{ \min_{2 \le j \le i} \{ j-1, m \} \right\} \cup \left\{ 1 + (i - 2 + t_1)\, \operatorname{sgn}\left(\max\{ m + 2 - t_1 - i, 0 \} \right) \right\}, & \text{if } t_1 < m \\ \left\{ \min_{2 \le j \le i} \{ j-1, m \} \right\}, & \text{if } t_1 \ge m \end{cases}$$

The following results are immediate consequences of theorem 3.1 for $t_1 = 1$ and unspecified m

3.1.1 Corollary on Analytic Determination of the Set of Feasible Ages at Each Stage with Starting Age 1

If the replacement age m is unspecified, set $m = \infty$. Then $S_j = \begin{cases} \{1, 2, \cdots, j-1\}, & \text{for } j \in \{2, 3, \cdots, m+1\} \\ \{1, 2, \cdots, m\}, & \text{for } j \in \{m+1, \cdots, n\} \end{cases}$

If $t_1 = 0$, then $S_i = \{1, \cdots, j-1\} \cup \{1 + t_1\}$, $i \in \{2, \cdots, n\}$

If the replacement age m is not specified, set $m = \infty$, in which case

If $t_1 = 0$, then $S_j = \{1, \cdots, j-1\}$, for $j \in \{2, \cdots, n\}$

An algorithm and solution template will now be designed, based on the author's theorem as formulated and proved below.

3.2 Theorem on Optimal Policy prescriptions Returns based on Multiple Starting Ages

Let the given problem be of horizon length n, with an arbitrary feasible starting age. Suppose that n_2 is another horizon length for the same problem such that the following conditions hold:.

(i) $n_2 \geq n$

(ii) $1+n_2-n \geq m+1$, where m is the equipment replacement age Then $D_j^*(s_j),\ f_j(s_j)$ are stage

(iii) The n_2- stage problem has the starting age 0, in stage 1 of the decision process.

j optimal decisions and rewards from the template with horizon length n_2, for $j \in \{n_2+1-n,\cdots,n_2\}$, with the set of starting ages $\{1,2,\cdots,m\}$ if and only if $D^*_{j+n-n_2}(s_{j+n-n_2})$ and $f_{j+n-n_2}(s_{j+n-n_2})$ are the corresponding optimal decisions and rewards in stage $j+n-n_2$ for the template with the horizon length n, with the same set of starting ages $\{1,2,\cdots,m\}$. Moreover, the optimal decisions and corresponding rewards for the base problem with starting age 0 are immediate from the choice $n_2 = n$, with condition (ii) waived.

Proof

Stage numbers do not feature explicitly in the solution; they feature implicitly from the fact that the feasible age states are functions of the stage number. To obtain the optimal strategies for an n-horizon problem from the corresponding n_2-horizon problem, $n_2 > n$, for multiple starting ages, simple solve the problem top down (backward dynamic programming approach) for n stages. Therefore the problem must be solved for stages $n_2,\ n_2-1,\cdots,x$ such that $1+n_2-x=n,\ \Rightarrow x = 1+n_2-n.$ Let S_i denote the set of feasible starting ages for the n-horizon problem, $i \in \{1,2,\cdots,n\}$. Then by defining $S_j = \hat{S}_{j+n-n_2}$, for $j \in \{1+n_2-n, 2+n_2-n,\cdots,n_2\}$, it is clear that $S_{1+n_2-n} = \hat{S}_1;\ S_{n_2} = \hat{S}_n$.

Hence $S_{1+n_2-n} = \hat{S}_1$ constitutes the set of feasible initial starting ages for the revised stage 1 of the n-horizon problem for the given n_2- horizon length, while $S_{n_2} = \hat{S}_n$ constitutes the set of feasible starting ages in stage n of the n-horizon problem for the given n_2-horizon length, $n_2 > n.$ The implication of these facts/revelations is that the optimal policy prescriptions and the corresponding returns for the n-horizon problem for all feasible nonzero starting ages $\{1,2,\cdots,m\}$ can be secured in one fell swoop by choosing n_2 such that $n_2 \geq m+n,$ and storing the value 0 for the single starting age t_1 of the n_2-stage problem, in the appropriate cell location in stage 1 of the Excel template (by an appeal to corollary 3.1.1 and Ukwu [10]). Then, by restricting the set S_j to $j \in \{1+n_2-n, 2+n_2-n,\cdots,n_2\}$, the optimal rewards for the n-stage problem from stage i to n are read off in stage $i+n_2-n$ of the n_2-stage problem, while the optimal strategies can be secured from stages $i+n_2-n$ to n_2. Hence $D_j^*(s_j),\ f_j(s_j)$ are stage j optimal decisions and rewards from the template with horizon n_2, for $j \in \{n_2+1-n,\cdots,n_2\}$, with the set of starting ages $\{1,2,\cdots,m\}$ if and only if $D^*_{j+n-n_2}(s_{j+n-n_2})$ and $f_{j+n-n_2}(s_{j+n-n_2})$ are the corresponding optimal decisions and reward in stage $j+n-n_2$ for the template with the horizon length n, with the same set of starting ages $\{1,2,\cdots,m\}$. If the starting age is 0, for the n-horizon base problem set $n_2 = n$, with condition (ii) waived to secure the optimal solutions and corresponding rewards for the n-horizon problem.

For simplicity take $n_2 = m+n$, the minimum horizon length for the n_2-stage problem for the realization of the full set of feasible nonzero starting ages $\{1,2,\cdots,m\}$, with respect to the n-stage problem. Then $1+n_2-n = m+1$ and stages $m+1,\cdots,n_2$ are the stages of the n_2-stage problem corresponding to the n-stage problem.

Henceforth, the starting age $\boldsymbol{t_1 = 0}$ in (iii) will be referred to as starting age index 0 or simply index 0.

3.3 An Algorithm for the Implementation of Theorem 3.2

Step 1: Design of Excel Template

TABLE I: Excel Spreadsheet Layout, Documentation, Data and Fixed Value Storage, Stage Numbering, Policy Prescriptions and Reward Automation.

	A	B	C	D	E	F	G	…	…	…	N
1	**ERP Solution Template**					n	**Starting Age**				
2	**Replacement Age** $m =$			m_val	**yrs.**	n_val	$t_1 = 0$				
3		**Given Data**			**Stage**	[=1]					
4		I **=**	I_val	***V(0)* =**		[= 2]					
5	**Age *t* (yrs.)**										
6	**Revenue: *r(t)* ($)**	$r(0)$	$r(1)$	$r(2)$	…						
7	**Mnt. cost, *c(t)* ($)**	$c(0)$	$c(1)$	$c(2)$	…						
8	**Salvage value, s(t)**		$s(1)$	$s(2)$	$s(3)$	…					
9	***K***										
10	***R***										
11	**Opt. value: *f*(*t*)**										
12	**Opt. Decision**										
13	**State**										
14											
15					**Stage**	[=3]					
16											
17											
18											
19											
20											
⋮											

Use Excel column A and other indicated cell references for identifiers and documentation, as shown above, in bold type font. Save the revenue data in Excel row 6, in contiguous cell locations, beginning from column B; save the maintenance cost data in Excel row 7, in contiguous cell locations, beginning from column B; save the salvage data in Excel row 8, in contiguous cell locations, beginning from column C; save the identifiers in remaining stages $n-1$ to 1 in the above table, using the Copy and Paste functionality. Consecutive stages should be separated by a blank row.

[=2] Under the decision R, save the fixed value $V(0) = r(0) - c(0) - I$ under the fixed cell reference $F$4, using the code: = $B$6-$B$7-$C$4, <ENTER>

Store m and n in the fixed (absolute) cell references $D$2, $F$2 respectively, and type the identifier $\boldsymbol{t_1 = 0}$ in cell location $G$2.

To automate the stage numbering, perform the following actions:

[=1]: Store last stage number n under the relative cell reference $F3, by typing: =$F$2 there, followed by <Enter>.

[=3]: Secure the stage number $n-1$ under the relative cell reference $F15, by typing: =$F$2 - 1 there, followed by <Enter>.

Secure the stage number $n-2$ under the relative cell reference $F22, by typing: =$F15 - 1 there, followed by <Enter>.

Step 2: Automation of the states in all n stages

Blank out column B, beginning from row 8.

Type the following code in C13:

= IF (B13 >= $D$2, "", IF (B13 = MIN ($F3-1, $D$2), "", 1+B13)) <Enter>.

Click back on cell C13 and position the cursor at the right edge of the cell until a crosshair appears. Then drag the crosshair across to the last the cell N13 to secure the stage n states with trailing blank spaces.

Henceforth, the act of clicking back on a specified cell, positioning the cursor at the right edge of the cell until a crosshair appears and the crosshair-dragging routine will be referred to as clerical routine/duty.

Now copy C13:N13 and paste it successively onto the cell references $\mathrm{C}\left[13+7\left(n-i\right)\right]:\mathrm{N}\left[13+7\left(n-i\right)\right]$, for $i \in \left\{n-1,\ n-2,\cdots,2,1\right\}$, to secure the states in the remaining $n-1$ stages.

Step 3: Stage *n* Computations

For $t = 1$, under REPLACE, type the following code in the cell reference C10:

=If (C13 = “”,””, $F$4+ $C$8+C$8) <ENTER> to secure $f_n^R(1)$.

Perform the horizontal clerical duty across to the last cell location N10, to secure

$f_n^R(s_n)$, and trailing blank spaces, for each $s_n \in S_n, s_n \geq 2$

For $t = 1$, under KEEP, type the following code in the cell reference C9:

=If ($C13 =$D$2,”Must Replace”, if (C13= “”,””, C$6-C$7+D$8)) <ENTER> to secure $f_n^K(1)$.

Perform the clerical duty to secure $f_n^K(s_n)$, and trailing blank spaces, for each $s_n \in S_n, s_n \geq 2$ .To secure $f_n(s_n)$, for $s_n \in S_n$, type the following code in the cell reference C11:

=If (C13 = “ ”,” ”,if (C9 = “Must Replace”, C10, max(C9,C10))) <ENTER> to secure $f_n(1)$.

Then perform the clerical routine across to N13 to secure $f_n(s_n), s_n \in S_n : s_n \geq 2$ and blank spaces

3.3.1 Remarks on Segment Code Redundancy

In Excel, the max and min functions return values for only numeric expressions, ignoring string constants; for example if the number 8 is saved in B2 and the string constant “**Must**” is saved in C2, Then in D2, the code: = max(B2, C2) <Enter> returns 8. In E2, the code: = max (B2, C2) <Enter> also returns 8. Therefore the code segment involving “if (C9 = “Must Replace”, C10” may be dispensed with throughout the template.

To obtain the optimal decision for each of the stage n states $s_n \in S_n$, type the following code in the cell reference C12:

=If (C13 = " "," ",if(C13 = $D$2, "R", if(C9 = C10, "K/R", if(C9 > C10, "K", "R")))) <ENTER> to secure $D_n(1)$.

Then perform the clerical routine to secure $D_n(s_n)$, for $s_n \in S_n : s_n \geq 2$ and blank spaces in sequence

Step 4: Stage (*n* - 1) Computations:

For $t = 1$, under REPLACE, type the following code in the cell reference C17:

=If (C20 = "","", $F$4+ C$8+$C11) <ENTER> to secure $f_{n-1}^{R}(1)$.

Perform the clerical duty to secure $f_{n-1}^{R}(s_{n-1})$, and trailing blank spaces, for each $s_{n-1} \in S_{n-1}, s_{n-1} \geq 2$ and succeeding blank spaces

For $t = 1$, under KEEP, type the following code in the cell reference C16:

=If (C20 =$D$2,"Must Replace", if (C20 = "","", C$6-C$7+D11)) <ENTER> to secure $f_7^K(1)$. Perform the clerical duty to secure $f_{n-1}^{K}(s_{n-1})$, and trailing blank spaces, for each $s_{n-1} \in S_{n-1}, s_{n-1} \geq 2$

To secure $f_{n-1}(s_{n-1})$, for $s_{n-1} \in S_{n-1}$ type the following code in the cell reference C18:

=If (C20 = " "," ", if (C16 = "Must Replace", C17, max(C16,C17)))<ENTER> to secure $f_{n-1}(1)$. Then perform the clerical routine to secure $f_{n-1}(s_{n-1})$, for $s_{n-1} \in S_{n-1}, s_{n-1} \geq 2$ and succeeding blank spaces.

To obtain the optimal decision for each of the stage $n-1$ states, type the following code in the cell reference C19:

=If (C20 = " "," ",if(C20 =$D$2, "R", if(C16 = C17, "K/R", if (C16 > C17, "K", "R"))))<ENTER> to secure $D_{n-1}(1)$.

Then perform the clerical routine to secure $D_n(s_n)$, for $s_n \in S_n : s_n \geq 2$ and trailing blanks

Step 5: Stage (*n* - 2) Computations

Copy the contiguous region $A15:N20 of stage *n*-1 into the contiguous region $A22:N27 of stage *n*– 2 to secure stage (*n* - 2) computational values

Step 6: Stage *i* Implementations, $i \in \{n-3, \cdots, 2, 1\}$, in One Fell Swoop

This is a crucial step involving a single Copy and $n-3$ Paste Operations, using the contiguous region $A22:N27 of stage $(n-2)$. Simply use the Copy and Paste functionality to copy and paste the contiguous region $A22:N27 successively into stages $(n-3)$ to 1 regions.

Note: Consecutive stages should be separated by a blank row. In other words, for $i \in \{n-3, n-4, \cdots, 1\}$ the Copy and Paste functionality can be used to copy and paste the contiguous region $A22:N27 successively into stages $(n-3)$ to 1 regions:

$$\text{A\$}\left[8+7(n-i)\right] : \text{A\$}\left[13+7(n-i)\right].$$

Step 7: Inputting the Starting Age $t_1 = 0$ in Stage 1

Simply type in the starting age 0 in cell $B [$6+7n$] <Enter> and then copy the contiguous region $\$C[2+7n]:\$C[5+7n]$ into the contiguous region $\$B[2+7n]:\$B[5+7n]$ to complete the computational process.

Note that the stage numbering is automatically implemented, computations in all stages are automatically executed and the problem with the starting age 0 is correctly solved right-off-the-bat.

Note that the starting age identifier $t_1 = 0$ indicates that $G$2 does not feature in the computations.

Step 8: Batch Solution Implementations for Feasible Starting Ages $\{1, 2, \cdots, m\}$, in One Fell Swoop, for an *n*-Stage Problem

Choose $n_2 : n_2 - n \geq m$. Indeed, without any loss of generality choose $n_2 = m + n$. Store the above value of n_2 in the fixed cell reference $F$2.

Use the template to implement the optimal solutions and the corresponding rewards for the n_2-stage problem. Go to stage $1+n_2-n$ of the template for the n_2 horizon problem.
Clearly $S_{1+n_2-n}=\{1,2,\cdots,m\}$, by an appeal to corollary 3.1.1.

The optimal policy prescriptions and corresponding rewards for the n- horizon problem for the entire set of feasible nonzero starting ages $\{1,2,\cdots,m\}$ from stage 1 to n are exactly the same as those of the n_2 horizon problem from stage $1+n_2-n$ to n_2. Awesome huh!
So, simply pick up the optimal policy prescriptions and corresponding rewards from there.

3.3.2 Remarks on the Use of the Templates for Large Problem Sizes

It is clear that the crosshair horizontal-dragging routine must be extended beyond column N, as appropriate, if $m\geq 13$. This can be optimally done before the Copy and Paste operations from stage $n-1$. Hence the template can be adequately appropriated for sensitivity analyses on this class of Equipment Replacement problems in just a matter of minutes, as contrasted with manual investigations that would at best consume hours or days with increasing values of m and/or n and the number of investigations, not to talk of the dire consequences of committing just one error in any stage computations.

3.3.3 Implication of the Algorithm

The implication of theorem 3.2 is that, for any problem instance with a given planning horizon length, n the optimal strategies and rewards for all corresponding problems of $2\leq$ horizon length $<n$ are automatically generated from the n_2- horizon solution template simultaneously for the set of feasible nonzero starting ages. Furthermore, the optimal strategies and rewards for the base problems with horizon length, n are generated from the n- horizon solution template for the zero starting age (new machine). In the sequel an application problem is given below to illustrate the solution template implementations.

3.4 Application Problems on Theorem 3.1 and the Implementation of the Solution Templates

A company needs to determine the optimal replacement policy for a current t_1- year old equipment over the next n years. The following table gives the data of the problem. The company requires that a 6 – year old equipment be replaced. The cost of a new machine is $100,000.

TABLE II: Pertinent Data for Optimal Policy and Reward Determination

Age: t yrs.	Revenue: $r(t)$ ($)	Operating cost: $c(t)$ ($)	Salvage value: $s(t)$ ($)
0	20,000	200	-
1	19,000	600	80,000
2	18,500	1,200	60,000
3	17,200	1,500	50,000
4	15,500	1,700	30,000
5	14,00	1,800	10,000
6	12,200	2,200	5,000

(a) Obtain the optimal policy prescriptions and the corresponding rewards in one fell swoop for $n\in\{2,3,\cdots,7\}$ and the set of starting ages $\{1,2,3,4,5,6\}$, using dynamic programming recursions, based on index 0.
(b) What minimum horizon length template is required to obtain the optimal policy prescriptions and the corresponding rewards in one fell swoop for the n- horizon problems, $n\in\{8,9,\cdots 12\}$ with the set of starting ages $\{1,2,3,4,5,6\}$, using dynamic programming recursions? Using this horizon length, obtain the optimal returns for problem (b) with respect to the set of starting age $\{1,2,3,4,5,6\}$.
(c) Extend the optimal policy prescriptions and corresponding rewards to a planning horizon of 13 years.

Solution

(a) In the given problem $m=6$. The maximum horizon length is 7. Therefore the minimum horizon length required for the solution template is $n_2: 1+n_2-n_{\max}=m+1 \Rightarrow n_2=m+n_{\max} \Rightarrow n_2=6+7=13$.

Equipment Replacement Problem Solution Template					n	Starting Age	
Replacement Age =			6	yrs	13	$t_1 = 0$	
	Given Data			Stage	13		
	I =	100000	$V(0) = r(0)-c(0) - I$ =		-80200		
Age t (yrs.)	0	1	2	3	4	5	6
Revenue: $r(t)$ ($)	20000	19000	18500	17200	15500	14000	12200
Mnt. cost, $c(t)$ ($)	200	600	1200	1500	1700	1800	2200
Salvage value, s(t)		80000	60000	50000	30000	10000	5000
K		78400	67300	45700	23800	17200	Must Replace
R		79800	59800	49800	29800	9800	4800
Opt. value: f(t)		79800	67300	49800	29800	17200	4800
Opt. Decision		*R*	*K*	*R*	*R*	*K*	*R*
State		1	2	3	4	5	6
				Stage	12		
K		85700	67100	45500	31000	17000	Must Replace
R		79600	59600	49600	29600	9600	4600
Opt. value: f(t)		85700	67100	49600	31000	17000	4600
Opt. Decision		*K*	*K*	*R*	*K*	*K*	*R*
State		1	2	3	4	5	6
				Stage	11		
K		85500	66900	46700	30800	16800	Must Replace
R		85500	65500	55500	35500	15500	10500
Opt. value: $f(t)$		85500	66900	55500	35500	16800	10500
Opt. Decision		K/R	K	R	R	K	R
State		1	2	3	4	5	6
				Stage	10		
K		85300	72800	51200	30600	22700	Must Replace
R		85300	65300	55300	35300	15300	10300
Opt. value: $f(t)$		85300	72800	55300	35300	22700	10300
Opt. Decision		*K/R*	*K*	*R*	*R*	K	R
State		1	2	3	4	5	6
				Stage	9		
K		91200	72600	51000	36500	22500	Must Replace
R		85100	65100	55100	35100	15100	10100
Opt. value: $f(t)$		91200	72600	55100	36500	22500	10100
Opt. Decision		*K*	*K*	*R*	*K*	K	R
State		1	2	3	4	5	6
				Stage	8		
K		91000	72400	52200	36300	22300	Must Replace
R		91000	71000	61000	41000	21000	16000
Opt. value: $f(t)$		91000	72400	61000	41000	22300	16000
Opt. Decision		*K/R*	*K*	*R*	R	K	R
State		1	2	3	4	5	6
				Stage	7		
K		90800	78300	56700	36100	28200	Must Replace
R		90800	70800	60800	40800	20800	15800
Opt. value: $f(t)$		90800	78300	60800	40800	28200	15800
Opt. Decision		*K/R*	*K*	R	R	K	R
State		1	2	3	4	5	6

Figure I: Template Solutions of the ERP for horizon length $j \in \{2, 3, \cdots, 7\}, \{t_1\} = \{1, 2, \cdots, 6\}$

Equipment Age Transition Diagrams for the Optimal policy Prescriptions corresponding to various Starting Ages for the 2-year horizon problem using the Decision and Salvage symbols *K, R, S*

These can be promptly obtained from stages 12 and 13 of the 13-stage problem. Simply use the stages 12 and 13, with the starting ages in stage 12, translating to the following equipment age transition diagrams and optimal values:

Age 1: $\mathbf{1}K\mathbf{2}K\mathbf{3}S \Rightarrow$ Unique optimum

Optimal value = (Maximum Net Income for years 1 and 2) $= \$\mathbf{85,700.00}$

Age 2: $2K\mathbf{3}R\mathbf{1}S \Rightarrow$ Unique optimum

Optimal value = (Maximum Net Income for years 1 and 2) $= \$\mathbf{67,100.00}$

Age 3: $\mathbf{3}R\mathbf{1}R\mathbf{1}S \Rightarrow$ Unique Optimum

Optimal value = (Maximum Net Income for years 1 and 2) $= \$\mathbf{49,600.00}$

Age 4: $4K5K6S \Rightarrow$ Unique Optimum

Optimal value = (Maximum Net Income for years 1 and 2) $= \$31,000.00$

Age 5: $5K6R1S \Rightarrow$ Unique Optimum

Optimal value = (Maximum Net Income for years 1 and 2) $= \$17,000.00$

Age 6: $6R1R1S \Rightarrow$ Unique Optimum

Optimal value = (Maximum Net Income for years 1 and 2) $= \$\mathbf{4,600.00}$

Optimal policy prescription corresponding to $1K2K3S$:

Start with a one-year machine at the beginning of decision year 1; keep (deploy) the machine for the next two years until the end of the decision year 2 when it is mandatorily salvaged. The net profit generated would be $85,700.00

Age Transition Diagrams for the Optimal policy Prescriptions corresponding to various Starting Ages for the 3-year horizon problem using the Decision and Salvage symbols *K, R, S*

These can be promptly obtained from stages 11 to 13 of the 13-stage problem. Simply use the stages 11 to 13, with the starting ages in stage 11, translating to the following equipment age transition diagrams and optimal values:

Age 1: $\mathbf{1}K\mathbf{2}K\mathbf{3}R\mathbf{1}S$; $\mathbf{1}R\mathbf{1}K\mathbf{2}K\mathbf{3}S \Rightarrow$ Alternate optima

Optimal value = (Maximum Net Income for years 1 to 3) $= \$\mathbf{85,500.00}$

Age 2: $2K\mathbf{3}R\mathbf{1}R\mathbf{1}S \Rightarrow$ Unique optimum

Optimal value = (Maximum Net Income for years 1 to 3) $= \$\mathbf{66,900.00}$

Age 3: $\mathbf{3}R\mathbf{1}K\mathbf{2}K\mathbf{3}S \Rightarrow$ Unique Optimum

Optimal value = (Maximum Net Income for years 1 to 3) $= \$\mathbf{55,500.00}$

Age 4: $4R1K2K3S \Rightarrow$ Unique Optimum

Optimal value = (Maximum Net Income for years 1 to 3) $= \$\mathbf{35,500.00}$

Age 5: $5K\mathbf{6}R\mathbf{1}R\mathbf{1}S \Rightarrow$ Unique Optimum

Optimal value = (Maximum Net Income for years 1 to 3) $= \$\mathbf{16,800.00}$

Age 6: $6R1K2K3S \Rightarrow$ Unique Optimum

Optimal value = (Maximum Net Income for years 1 to 3) $= \$\mathbf{10,500.00}$

Interpretation of a selected transition diagram

$1K2K3R1S$: Machine Age 1 transits to 2 and 3 after the machine has been deployed for 1 and 2 years respectively. Then the three year-old machine is replaced and deployed for one year, whereupon the age of the machine becomes 1 at the end of year 3, noting that the age of the replacement machine at the beginning of year 3 is 0.

Optimal policy prescription corresponding to $1K2K3R1S$:

Start with a one-year machine at the beginning of decision year 1; keep (deploy) the machine for the next two years and then replace it at the beginning of the decision year 3 until the end of the decision year 3 when it is mandatorily salvaged.

Age Transition Diagrams for the Optimal Policy Prescriptions corresponding to various Starting Ages for the 4-year horizon problem using the Decision and Salvage symbols *K, R, S*

Age 1: $\mathbf{1}K\mathbf{2}K\mathbf{3}R\mathbf{1}R\mathbf{1}S$; $\mathbf{1}R\mathbf{1}K\mathbf{2}K\mathbf{3}R\mathbf{1}S$; $\mathbf{1}R\mathbf{1}R\mathbf{1}K\mathbf{2}K\mathbf{3}S \Rightarrow$ Alternate optima

Optimal value = (Maximum Net Income for years 1 to 4) = $\$\mathbf{85,300.00}$

Age 2: $2K\mathbf{3}R\mathbf{1}K\mathbf{2}K\mathbf{3}S \Rightarrow$ Unique optimum

Optimal value = (Maximum Net Income for years 1 to 4) = $\$\mathbf{72,800.00}$

Age 3: 3R1K2K3R1S;3R1R1K2K3S $\Rightarrow$ Alternate Optima

Optimal value = (Maximum Net Income for years 1 to 4) = $\$\mathbf{55,300.00}$

Age 4: $4R1K2K3R1S$;$4R1R1K2K3S \Rightarrow$ Alternate Optima

Optimal value = (Maximum Net Income for years 1 to 4) = $\$\mathbf{35,300.00}$

Age 5: 5K6R1K2K3S $\Rightarrow$ Unique Optimum

Optimal value = (Maximum Net Income for years 1 to 4) = $\$\mathbf{22,700.00}$

Age 6: $6R1K2K3R1S$;$6R1R1K2K3S \Rightarrow$ Alternate Optima

Optimal value = (Maximum Net Income for years 1 to 4) = $\$\mathbf{10,300.00}$

Age Transition Diagrams for the Optimal policy Prescriptions corresponding to various Starting Ages for the 5-year horizon problem using the Decision and Salvage symbols *K, R, S*

Start from stage 9, secure the three concatenated objects for each starting age and proceed to the relevant starting age in stage 10, to complete the transition diagrams:

$1K2 \rightarrow$ Concatenate $1K$ with starting age 2 transition diagrams starting from stage 10 up

$2K3 \rightarrow$ Concatenate $2K$ with starting age 3 transition diagrams starting from stage 10 up

$3R1 \rightarrow$ Concatenate $3R$ with starting age 1 transition diagram starting from stage 10 up

$4K5 \rightarrow$ Concatenate $4K$ with starting age 5 transition diagrams starting from stage 10 up

$5K6 \rightarrow$ Concatenate $5K$ with starting age 6 transition diagrams starting from stage 10 up

$6R1 \rightarrow$ Concatenate $6R$ with starting age 1 transition diagrams starting from stage 10 up

Hence

Age 1: $1K2K\mathbf{3}R\mathbf{1}K\mathbf{2}K\mathbf{3}S \Rightarrow$ Unique optimum

Optimal value = (Maximum Net Income for years 1 to 5) = $\$\mathbf{91,200.00}$

Age 2:$2K$3R1K2K3R1S;$2K$3R1R1K2K3S $\Rightarrow$ Alternate Optima

Optimal value = (Maximum Net Income for years 1 to 5) = $\$\mathbf{72,600.00}$

Age 3: $3R\mathbf{1}K\mathbf{2}K\mathbf{3}R\mathbf{1}R\mathbf{1}S$; $3R\mathbf{1}R\mathbf{1}K\mathbf{2}K\mathbf{3}R\mathbf{1}S$; $3R\mathbf{1}R\mathbf{1}R\mathbf{1}K\mathbf{2}K\mathbf{3}S \Rightarrow$ Alternate optima

Optimal value = (Maximum Net Income for years 1 to 5) = $\$\mathbf{55,100.00}$

Age 4: 4K5K6R1K2K3SÞUnique Optimum

Optimal value=(Maximum Net Income for years 1 to 5) =$36,500.00

Age 5: $5K6R1K2K3R1S$;$5K6R1R1K2K3S \Rightarrow$ Alternate Optima

Optimal value = (Maximum Net Income for years 1 to 5) = $\$\mathbf{22,500.00}$

Age 6: $6R\mathbf{1}K\mathbf{2}K\mathbf{3}R\mathbf{1}R\mathbf{1}S$; $6R\mathbf{1}R\mathbf{1}K\mathbf{2}K\mathbf{3}R\mathbf{1}S$; $6R\mathbf{1}R\mathbf{1}R\mathbf{1}K\mathbf{2}K\mathbf{3}S \Rightarrow$ Alternate optima

Optimal value = (Maximum Net Income for years 1 to 5) $=\$\mathbf{10,100.00}$

Age Transition Diagrams for the Optimal policy Prescriptions corresponding to various Starting Ages for the 6-year horizon problem using the Decision and Salvage symbols *K, R, S*

Start from stage 8, secure the three concatenated objects for each starting age and proceed to the relevant starting age in stage 9, to complete the equipment age transition diagrams:

$1K2; 1R1 \rightarrow$ Concatenate $1K$ and $1R$ with starting ages 2 and 1 transition diagrams respectively

from stage 9 up

$2K3 \rightarrow$ Concatenate $2K$ with starting age 3 transition diagrams from stage 9 up

$3R1 \rightarrow$ Concatenate $3R$ with starting age 1 transition diagram from stage 9 up

$4R1 \rightarrow$ Concatenate $4R$ with starting age 1 transition diagrams from stage 9 up

$5K6 \rightarrow$ Concatenate $5K$ with starting age 6 transition diagrams from stage 9 up

$6R1 \rightarrow$ Concatenate $6R$ with starting age 1 transition diagrams from stage 9 up

Age 1: $1K\mathbf{2}K\mathbf{3}R\mathbf{1}K\mathbf{2}K\mathbf{3}R\mathbf{1}S$; $\mathbf{1}K\mathbf{2}K\mathbf{3}R\mathbf{1}R\mathbf{1}K\mathbf{2}K\mathbf{3}S$; $1R1K2K\mathbf{3}R\mathbf{1}K\mathbf{2}K\mathbf{3}S \Rightarrow$ Alternate Optima

Optimal value = (Maximum Net Income for years 1 to 6) $=\$\mathbf{91,000.00}$

Age 2: $2K3R\mathbf{1}K\mathbf{2}K\mathbf{3}R\mathbf{1}R\mathbf{1}S$; $2K3R\mathbf{1}R\mathbf{1}K\mathbf{2}K\mathbf{3}R\mathbf{1}S$; $2K3R\mathbf{1}R\mathbf{1}R\mathbf{1}K\mathbf{2}K\mathbf{3}S \Rightarrow$ Alternate optima

Optimal value = (Maximum Net Income for years 1 to 6) $=\$\mathbf{72,400.00}$

Age 3: $3R1K2K\mathbf{3}R\mathbf{1}K\mathbf{2}K\mathbf{3}S \Rightarrow$ Unique optimum

Optimal value = (Maximum Net Income for years 1 to 6) $=\$\mathbf{61,000.00}$

Age 4: $4R1K2K\mathbf{3}R\mathbf{1}K\mathbf{2}K\mathbf{3}S \Rightarrow$ Unique optimum

Optimal value = (Maximum Net Income for years 1 to 6) $=\$\mathbf{41,000.00}$

Age 5: $5K6R\mathbf{1}K\mathbf{2}K\mathbf{3}R\mathbf{1}R\mathbf{1}S$; $5K6R\mathbf{1}R\mathbf{1}K\mathbf{2}K\mathbf{3}R\mathbf{1}S$; $5K6R\mathbf{1}R\mathbf{1}R\mathbf{1}K\mathbf{2}K\mathbf{3}S \Rightarrow$ Alternate optima

Optimal value = (Maximum Net Income for years 1 to 6) $=\$\mathbf{22,300.00}$

Age 6: $6R1K2K\mathbf{3}R\mathbf{1}K\mathbf{2}K\mathbf{3}S \Rightarrow$ Unique optimum

Optimal value = (Maximum Net Income for years 1 to 6) $=\$\mathbf{16,000.00}$

Age Transition Diagrams for the Optimal policy Prescriptions corresponding to various Starting Ages for the 7-year horizon problem using the Decision and Salvage symbols *K, R, S*

Start from stage 7, secure the three concatenated objects for each starting age and proceed to the relevant starting age in stage 8, to complete the transition diagrams:

$1K2; 1R1 \rightarrow$ Concatenate $1K$ and $1R$ with starting ages 2 and 1 transition diagrams respectively

from stage 8 up

$2K3 \rightarrow$ Concatenate $2K$ with starting age 3 transition diagrams from stage 8 up

$3R1 \rightarrow$ Concatenate $3R$ with starting age 1 transition diagrams from stage 8 up

$4R1 \rightarrow$ Concatenate $4R$ with starting age 1 transition diagrams from stage 8 up

$5K6 \rightarrow$ Concatenate $5K$ with starting age 6 transition diagrams from stage 8 up

$6R1 \rightarrow$ Concatenate $6R$ with starting age 1 transition diagrams from stage 8 up

Age 1: $1K2K3R1K2K3R1R1S$; $1K2K3R1R1K2K3R1S$; $1K2K3R1R1R1K2K3S$;

$1R1K2K3R1K2K3R1S$; $1R1K2K3R1R1K2K3S$; $1R1R1K2K3R1K2K3S$

$\Rightarrow$ Alternate Optima

Optimal value = (Maximum Net Income for years 1 to 7) = $\$\mathbf{90,800.00}$

Age 2: $2K3R1K2K3R1K2K3S \Rightarrow$ Unique optimum

Optimal value = (Maximum Net Income for years 1 to 7) = $\$\mathbf{78,300.00}$

Age 3: $3R1K2K3R1K2K3R1S$; $3R1K2K3R1R1K2K3S$; $3R1R1K2K3R1K2K3S$

$\Rightarrow$ Alternate Optima

Optimal value = (Maximum Net Income for years 1 to 7) = $\$\mathbf{60,800.00}$

Age 4: $4R1K2K3R1K2K3R1S$; $4R1K2K3R1R1K2K3S$; $4R1R1K2K3R1K2K3S$

$\Rightarrow$ Alternate Optima

Optimal value = (Maximum Net Income for years 1 to 7) = $\$\mathbf{40,800.00}$

Age 5: $5K6R1K2K3R1K2K3S \Rightarrow$ Unique optimum

Optimal value = (Maximum Net Income for years 1 to 7) = $\$\mathbf{28,200.00}$

Age 6: $6R1K2K3R1K2K3R1S$; $6R1K2K3R1R1K2K3S$; $6R1R1K2K3R1K2K3S$

$\Rightarrow$ Alternate Optima

Optimal value = (Maximum Net Income for years 1 to 7) = $\$\mathbf{15,800.00}$

For the problems of horizon length $n \in \{8, 9, \cdots, 12\}$, by invoking corollary 3.1.1 and theorem 3.2, the starting set of ages is $\{1, \cdots, 13-n\} \subseteq \{1, \cdots, 5\} \subset \{1, \cdots, 6\}$. To get the full set of nonzero starting ages choose a horizon length $n_3 > 12 : n_3 - 12 \geq 6$. Without any loss of generality take $n_3 = 18$. So the template for the n- stage problem starts from stage $1 + n_3 - n = 19 - n$ of the 18-stage problem. The global results are as tabulated below.

The templates for starting age 0 are furnished below for horizon lengths $n \in \{1, 2, \cdots, 7\}$, followed by the age transition diagrams and the corresponding optimal returns.

Equipment Replacement Problem Solution Template					n	Starting Age	
Replacement Age =			6	**yrs**	1	Zero	
	Given Data			**Stage**	1		
	I **=**	100000	***V(0) = r(0)-c(0) - I =***		-80200		
Age *t* (yrs.)	0	1	2	3	4	5	6
Revenue: *r(t)* ($)	20000	19000	18500	17200	15500	14000	12200
Mnt. cost, *c(t)* ($)	200	600	1200	1500	1700	1800	2200
Salvage value, s(t)		80000	60000	50000	30000	10000	5000
K	99800						
R	-200						
Opt. value: f(t)	99800						
Opt. Decision	K						
State	0						

Figure II: Template Solutions of the Equipment Replacement Problem for the 1 - year horizon problem, with respect to the Starting Age

Equipment Replacement Problem Solution Template					*n*	**Starting Age**	
Replacement Age =			6	**yrs**	2	Zero	
	Given Data			**Stage**	2		
	I **=**	100000	***V(0) = r(0)-c(0) - I =***		-80200		
Age *t* (yrs.)	0	1	2	3	4	5	6
Revenue: *r(t)* ($)	20000	19000	18500	17200	15500	14000	12200
Mnt. cost, *c(t)* ($)	200	600	1200	1500	1700	1800	2200
Salvage value, s(t)		80000	60000	50000	30000	10000	5000
K		78400					
R		79800					
Opt. value: f(t)		79800					
Opt. Decision		*R*					
State		1					
				Stage	1		
K	99600						
R	-400						
Opt. value: f(t)	99600						
Opt. Decision	*K*						
State	0						

Figure III: Template Solutions of the Equipment Replacement Problem for the 2 - year horizon problem, with respect to the Starting Age 0

Equipment Replacement Problem Solution Template					*n*	**Starting Age**	
Replacement Age =			6	**yrs**	3	Zero	
	Given Data			**Stage**	3		
	I **=**	100000	***V(0) = r(0)-c(0) - I =***		-80200		
Age *t* (yrs.)	0	1	2	3	4	5	6
Revenue: *r(t)* ($)	20000	19000	18500	17200	15500	14000	12200
Mnt. cost, *c(t)* ($)	200	600	1200	1500	1700	1800	2200
Salvage value, s(t)		80000	60000	50000	30000	10000	5000
K		78400	67300				
R		79800	59800				
Opt. value: f(t)		79800	67300				
Opt. Decision		*R*	*K*				
State		1	2				
				Stage	2		
K		85700					
R		79600					
Opt. value: f(t)		85700					
Opt. Decision		*K*					
State		1					
				Stage	1		
K	105500						
R	5500						
Opt. value: *f(t)*	105500						
Opt. Decision	*K*						
State	0						

Figure IV: Template Solutions of the Equipment Replacement Problem for the 3 - year horizon problem, with respect to the Starting Age 0

Equipment Replacement Problem Solution Template					n	**Starting Age**	
Replacement Age =			6	**yrs**	4	Zero	
	Given Data			**Stage**	4		
	***I* =**	100000	***V(0) = r(0)-c(0) - I =***		-80200		
Age *t* (yrs.)	0	1	2	3	4	5	6
Revenue: *r(t)* ($)	20000	19000	18500	17200	15500	14000	12200
Mnt. cost, *c(t)* ($)	200	600	1200	1500	1700	1800	2200
Salvage value, s(t)		80000	60000	50000	30000	10000	5000
K		78400	67300	45700			
R		79800	59800	49800			
Opt. value: f(t)		79800	67300	49800			
Opt. Decision		*R*	*K*	*R*			
State		1	2	3			
				Stage	3		
K		85700	67100				
R		79600	59600				
Opt. value: f(t)		85700	67100				
Opt. Decision		*K*	*K*				
State		1	2				
				Stage	2		
K		85500					
R		85500					
Opt. value: *f(t)*		85500					
Opt. Decision		K/R					
State		1					
				Stage	1		
K	105300						
R	5300						
Opt. value: *f(t)*	105300						
Opt. Decision	K						
State	0						

Figure V: Template Solutions of the Equipment Replacement Problem for the 4 - year horizon problem, with respect to the Starting Age 0

Equipment Replacement Problem Solution Template					*n*	**Starting Age**	
Replacement Age =			6	**yrs**	5	Zero	
	Given Data			**Stage**	5		
	I **=**	100000	***V(0) = r(0)-c(0) - I =***		-80200		
Age *t* (yrs.)	0	1	2	3	4	5	6
Revenue: *r(t)* ($)	20000	19000	18500	17200	15500	14000	12200
Mnt. cost, *c(t)* ($)	200	600	1200	1500	1700	1800	2200
Salvage value, s(t)		80000	60000	50000	30000	10000	5000
K		78400	67300	45700	23800		
R		79800	59800	49800	29800		
Opt. value: f(t)		79800	67300	49800	29800		
Opt. Decision		*R*	*K*	*R*	*R*		
State		1	2	3	4		
				Stage	4		
K		85700	67100	45500			
R		79600	59600	49600			
Opt. value: f(t)		85700	67100	49600			
Opt. Decision		*K*	*K*	*R*			
State		1	2	3			
				Stage	3		
K		85500	66900				
R		85500	65500				
Opt. value: *f(t)*		85500	66900				
Opt. Decision		K/R	K				
State		1	2				
				Stage	2		
K		85300					
R		85300					
Opt. value: *f(t)*		85300					
Opt. Decision		K/R					
State		1					
				Stage	1		
K	105100						
R	5100						
Opt. value: *f(t)*	105100						
Opt. Decision	K						
State	0						

Figure VI: Template Solutions of the Equipment Replacement Problem for the 5 - year horizon problem, with respect to the Starting Age 0

Equipment Replacement Problem Solution Template					*n*	**Starting Age**	
Replacement Age =			6	**yrs**	6	Zero	
	Given Data			**Stage**	6		
	I =	100000	***V(0) = r(0)-c(0) - I =***		-80200		
Age *t* (yrs.)	0	1	2	3	4	5	6
Revenue: *r(t)* ($)	20000	19000	18500	17200	15500	14000	12200
Mnt. cost, *c(t)* ($)	200	600	1200	1500	1700	1800	2200
Salvage value, s(t)		80000	60000	50000	30000	10000	5000
K		78400	67300	45700	23800	17200	
R		79800	59800	49800	29800	9800	
Opt. value: f(t)		79800	67300	49800	29800	17200	
Opt. Decision		*R*	*K*	*R*	*R*	*K*	
State		1	2	3	4	5	
				Stage	5		
K		85700	67100	45500	31000		
R		79600	59600	49600	29600		
Opt. value: f(t)		85700	67100	49600	31000		
Opt. Decision		*K*	*K*	*R*	*K*		
State		1	2	3	4		
				Stage	4		
K		85500	66900	46700			
R		85500	65500	55500			
Opt. value: *f(t)*		85500	66900	55500			
Opt. Decision		K/R	K	R			
State		1	2	3			
				Stage	3		
K		85300	72800				
R		85300	65300				
Opt. value: *f(t)*		85300	72800				
Opt. Decision		K/R	K				
State		1	2				
				Stage	2		
K		91200					
R		85100					
Opt. value: *f(t)*		91200					
Opt. Decision		K					
State		1					
				Stage	1		
K	111000						
R	11000						
Opt. value: *f(t)*	111000						
Opt. Decision	K						
State	0						

Figure VII: Template Solutions of the Equipment Replacement Problem for the 6 - year horizon problem, with respect to the Starting Age 0

Equipment Replacement Problem Solution Template					*n*	**Starting Age**	
Replacement Age =			6	**yrs**	7	Zero	
	Given Data			**Stage**	7		
	I =	100000	***V(0) = r(0)-c(0) - I =***		-80200		
Age *t* (yrs.)	0	1	2	3	4	5	6
Revenue: *r(t)* ($)	20000	19000	18500	17200	15500	14000	12200
Mnt. cost, *c(t)* ($)	200	600	1200	1500	1700	1800	2200
Salvage value, s(t)		80000	60000	50000	30000	10000	5000
K		78400	67300	45700	23800	17200	Must Replace
R		79800	59800	49800	29800	9800	4800
Opt. value: f(t)		79800	67300	49800	29800	17200	4800
Opt. Decision		*R*	*K*	*R*	*R*	*K*	*R*
State		1	2	3	4	5	6
				Stage	6		
K		85700	67100	45500	31000	17000	
R		79600	59600	49600	29600	9600	
Opt. value: f(t)		85700	67100	49600	31000	17000	
Opt. Decision		*K*	*K*	*R*	*K*	*K*	
State		1	2	3	4	5	
				Stage	5		
K		85500	66900	46700	30800		
R		85500	65500	55500	35500		
Opt. value: *f(t)*		85500	66900	55500	35500		
Opt. Decision		K/R	K	R	R		
State		1	2	3	4		
				Stage	4		
K		85300	72800	51200			
R		85300	65300	55300			
Opt. value: *f(t)*		85300	72800	55300			
Opt. Decision		K/R	K	R			
State		1	2	3			
				Stage	3		
K		91200	72600				
R		85100	65100				
Opt. value: *f(t)*		91200	72600				
Opt. Decision		K	K				
State		1	2				
				Stage	2		
K		91000					
R		91000					
Opt. value: *f(t)*		91000					
Opt. Decision		K/R					
State		1					
				Stage	1		
K	110800						
R	10800						
Opt. value: *f(t)*	110800						
Opt. Decision	K						
State	0						

Figure VIII: Template Solutions of the Equipment Replacement Problem for the 7 - year horizon problem, with respect to the Starting Age 0

TABLE III: Age Transition Diagrams for the Optimal policy Prescriptions corresponding to Starting Age 0 for the *n*-year horizon problem using the Decision and Salvage symbols *K, R, S*

n	Age Transition Diagrams	Optimal Returns($)
1	0*K*1*S*	99,800.00
2	*OK*1*R*1*S*	99,600.00
3	O*K*1*K*2*K*3*S*	105,500.00
4	O*K*1*K*2*K*3*R*1*S; O*K*1*R*1*K*2*K*3*S*	105,300.00
5	O*K*1*K*2*K*3*R*1*R*1*S; O*K*1*R*1*K*2*K*3*R*1*S; O*K*1*R*1*R*1*K*2*K*3*S*	105,100.00
6	O*K*1*K*2*K*3*R*1*K*2*K*3S	111,000.00
7	O*K*1*K*2*K*3*R*1*K*2*K*3*R*1*S; O*K*1*K*2*K*3*R*1*R*1*K*2*K*3*S; O*K*1*R*1*K*2*K*3*R*1*K*2*K*3*S*	110,800.00

TABLE IV: Summary of the Optimal Returns for Problems with Horizon Lengths $\{1, 2, \cdots, 13\}$, from Year 1 To the End of the Planning Horizon, For All Feasible Equipment Ages

Starting Ages →	**0**	**1**	**2**	**3**	**4**	**5**	**6**
Horizon lengths ↓	**Optimal Returns ($): Maximum Net Incomes**						
1	99,800.00	79,800.00	67,300.00	49,800.00	29,800.00	17,200.00	4,800.00
2	99,600.00	85,700.00	67,100.00	49,600.00	31,000.00	17,000.00	4,600.00
3	105,500.00	85,500.00	66,900.00	55,500.00	35,500.00	16,800.00	10,500.00
4	105,300.00	85,300.00	72,800.00	55,300.00	35,300.00	22,700.00	10,300.00
5	105,100.00	91,200.00	72,600.00	55,100.00	36,500.00	22,500.00	10,100.00
6	111,000.00	91,000.00	92,400.00	61,000.00	41,000.00	22,300.00	16,000.00
7	110,800.00	90,800.00	78,300.00	60,800.00	40,800.00	28,200.00	15,800.00
8	110,600.00	96,700.00	78,100.00	60,600.00	42,000.00	28,000.00	15,600.00
9	116,500.00	96,500.00	77,900.00	66,500.00	46,500.00	27,800.00	21,500.00
10	116,300.00	96,300.00	83,800.00	66,300.00	46,300.00	33,700.00	21,300.00
11	116,100.00	102,200.00	83,600.00	66,100.00	47,500.00	33,500.00	21,100.00
12	122,000.00	102,000.00	83,400.00	72,000.00	52,000.00	33,3000.00	27,000.00
13	121,800.00	101,800.00	89,300.00	71,800.00	51,800.00	39,200.00	26,800.00

The optimal prescriptions and rewards for the horizon length 13 problem were secured from $n_2 = 19,$ in stage s 7 to 19. The results are consistent with those of the 18-stage problem for the set of starting ages {1, 2, 3, 4, 5}, from stage 6 to stage 18. The horizon length $n_2 = 19$ was needed only to secure the results with respect to the starting age 6.

Note that for any horizon length the maximum net profit is a decreasing function of equipment starting age.

4. CONCLUSION

The article designed and automated prototypical solution templates for batch optimal policy prescriptions for a certain stationary class of equipment replacement problems, with any set of feasible starting ages, complete with an algorithmic exposition on the interface and solution process. The optimality results were assured by the appropriation of the structure of the set of feasible ages at each stage, a robust investigation of the solution templates in Ukwu [14] for the equipment starting age of 0with respect to the same problem but with longer horizon lengths, and by deft reasoning regarding the non-explicit dependence of the dynamic programming recursions on stage numbers. Finally the article deployed the template and decision symbols to obtain alternate batch optimal policy prescriptions with respect to relevant problems, with horizon lengths of 1 to13 years, and the full set of starting ages. These trail-blazing findings provide amazing and refreshing simplification perspectives, as well as a new paradigm on simultaneous generation of optimal strategies and returns for the given class of equipment replacement problems with any desired feasible batches of starting ages. Consequently, relevant multiple practical problems of any conceivable size can now be solved in just a matter of minutes as soon as the pertinent data have been organized and stored at the appropriate Excel cell locations, resulting in tremendous savings in time, cost and energy. Furthermore, any desired levels of sensitivity analyses can be easily undertaken and accomplished with great rapidity; needless to say that large-scale equipment replacement problems that hitherto could hardly be contemplated due to the 'curse of dimensionality' have now been reduced to 'a child's play.' The equipment age transition diagrams were leveraged on, as they provided veritable platforms for the interpretations of the optimal policy prescriptions. The efficiency, power and utility of the results are quite easily demonstrable, 'absolutely, positively, without a doubt.'

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