International Journal of Multidisciplinary Research and Development
Online ISSN: 2349-4182, Print ISSN: 2349-5979
Impact Factor: RJIF 5.72
www.allsubjectjournal.com
Volume 5; Issue 4; April 2018; Page No. 236-241

A block procedure with continuous coefficients for the direct solution of general second order initial value problems of (ODEs) using shifted Legendre polynomials as basis function

Kamoh NM[1], Abada AA[2], Soomiyol Mrumun Comfort[3]
[1] Department of Mathematics/Statistics, Bingham University, Karu, Keffi-Abuja Expressway, Nigeria
[2] Department of Mathematics/Statistics, Taraba State University, Jalingo, Nigeria
[3] Department of Mathematics/Computer Science, Benue State University, Makurdi, Nigeria

Abstract
This paper presents a self starting block method for the direct solution of general second order initial value problems of ordinary differential equations. The method was developed via interpolation and collocation of the shifted Legendre polynomial as basis function. A continuous linear multistep method was generated and was evaluated at some desired points to give the discrete block method. The block method was investigated and was found to be consistent, zero stable and convergent. The method was applied on some nonlinear as well as linear ordinary differential equations problems and the performance was relatively better than those constructed by [1], [12] and [15] respectively.

Keywords: collocation, interpolation, shifted legendre polynomials, block method, discrete method, consistent, zero stable, convergent

1. Introduction

The desire to obtain more accurate approximate solutions to mathematical models, arising from science, engineering and even social sciences, in the form of ordinary differential equations (ODEs), some of these include mechanical systems with several springs attached in series, series circuit [14]. It is a well known fact that most of these problems cannot be solved using analytical approach; hence these have led many scholars to propose several different numerical methods.

Over the years, several researchers have considered several techniques of generating numerical solutions of second order initial value problems for ordinary differential equations of the form

$$y'' = f(x, y, y'), y(0) = y_0, y'(0) = \beta \qquad (1)$$

There are currently two well known techniques for solving (1). The first is to reduce (1) to a system of first order ordinary differential equation and then solve using predictor corrector or Runge-Kutta method. The second approach is to solve (1) directly using the block method since it preserves the traditional advantage of one step methods, of being self-starting and permitting easy change of step length [7]. Its advantage over Runge-Kutta methods lies in the fact that they are less expensive in terms of the number of functions evaluation for a given order. The method generates simultaneous solutions at all grid points as investigated by researchers. A class of continuous linear multistep method for general second order initial value problems of ordinary differential equations was developed by [2]. A block method for the solution of second order ordinary differential equations was adopted by [3], [4], [5] and [19]. A family of implicit uniformly accurate order block integrators for the solution of second order differential equations was developed by [1], [18] and [20]. A new derivation of continuous multistep methods using power series as basis function was investigated by [9]. [15] and [16] proposed five-step and four-step self-starting methods which adopt continuous linear multistep method to obtain finite difference methods applied respectively as a block for the direct solution of second order ordinary differential equations. A robust optimal order formula for direct integration of second order orbital problems was developed by [13]. Similarly, [12] investigated a uniform order legendre approach for continuous hybrid block methods for the solution of first order ordinary differential equations and much recently [10] developed the numerical solution of third order ordinary differential equations using a seven-step block method. This approach solves higher order initial value problems of Ordinary differential equations (ODEs) without first going through the process of reduction which is known to have many setbacks; among these is wastage of computer time and human effort.

In this paper, we propose continuous block method. This continuity properties enable us to evaluate at all points within the interval of integration, hence enable the study of the dynamical system at all the grid points. The derivation of the block method of solution which is self starting and does not require any previous values is obtained simultaneously. In generating the continuous linear multistep method via interpolation and collocation approach, shifted Legendre polynomial is used as basis function.

2. Derivation of the method

Consider an approximate solution of (1) presented by the shifted Legendre polynomials of degree m of the form;

$$y(t) = \sum_{i=0}^{m} c_i p_i(t) \qquad (2)$$

where c_i∈R, yϵC^2 (a,b).
The second derivative of (2) gives

$$y'' = \sum_{i=0}^{m} c_i p_i''(t) \qquad (3)$$

Substituting (3) into (1) gives

$$y'' = \sum_{i=0}^{m} c_i p_i''(t) = f(x, y(x), y'(x)) \qquad (4)$$

Evaluating (4) at x_(n+r),r=(1)k and (2) at x_n and x_(n+k-1) respectively; gives a system of nonlinear algebraic equations of the form

$$AX = B \qquad (5)$$

where

$$A = \begin{bmatrix} p_0(0) & p_1(0) & p_2(0) & \cdots & p_m(0) \\ p_0((k-1)h) & p_1((k-1)h) & p_2((k-1)h) & \cdots & p_m((k-1)h) \\ p_0''(h) & p_1''(h) & p_2''(h) & \cdots & p_m''(h) \\ p_0''(2h) & p_1''(2h) & p_2''(h) & \cdots & p_m''(2h) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ p_0''((k-1)h) & p_1((k-1)h) & p_2((k-1)h) & \cdots & p_m((k-1)h) \\ p_0''(kh) & p_1''(kh) & p_0''(kh) & \cdots & p_m''(kh) \end{bmatrix} \quad X = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ \vdots \\ \vdots \\ c_{m-1} \\ c_m \end{bmatrix}, B = \begin{bmatrix} y_n \\ y_{n+k-1} \\ f_{n+1} \\ f_{n+2} \\ \vdots \\ \vdots \\ f_{n+k-1} \\ f_{n+k} \end{bmatrix} \qquad (6)$$

Solving for c_i's,i=0(1)m in (5) using inverse of a matrix method which are then substituted into (2) to produce a continuous implicit method;

$$y(x) = \sum_{j=0}^{k-1} \alpha_j(x) y_{n+j} + h^2 \sum_{j=0}^{k} \beta_j(x) f_{n+j} \qquad (7)$$

Specification of the method

Considering k=5, the interpolation of (2) at the points x_(n+r),r=0,4 and collocating (4) at the points x_(n+r),r= 1,2, 3, 4 and 5, solving for the c_i's and substituting in (2), leads to the continuous linear multistep method of the form

$$\left. Y(x) = \sum_{i=0}^{4} \alpha_i\,(x) y_{n+i} + h^2 \sum_{i=0}^{5} \beta_i\,(x) f_{n+i} \right\} \tag{8}$$

Where

$$\left[\begin{array}{c}
\alpha_0(x) = 1 - \frac{1}{4}t \\[2mm]
\alpha_4(x) = 1 - \frac{1}{4}t \\[2mm]
\beta_0(x) = \frac{1}{2}t^2 - \frac{137}{360h}t^3 + \frac{5}{32h^2}t^4 - \frac{17}{480h^3}t^5 + \frac{1}{240h^4}t^6 - \frac{1}{5040h^5}t^7 - \frac{94}{315}ht \\[2mm]
\beta_1(x) = \frac{5}{6h}t^3 - \frac{77}{144h^2}t^4 + \frac{71}{450h^3}t^5 - \frac{7}{360h^4}t^6 + \frac{1}{1008h^5}t^7 - \frac{356}{315}ht \\[2mm]
\beta_2(x) = \frac{13}{360h^4}t^6 + \frac{107}{144h^2}t^4 - \frac{5}{6h}t^3 - \frac{1}{504h^5}t^7 - \frac{44}{315}ht - \frac{59}{315h^3}t^5 \\[2mm]
\beta_3(x) = \frac{5}{9h}t^3 - \frac{13}{24h^2}t^4 + \frac{49}{240h^3}t^5 - \frac{1}{360h^4}t^6 + \frac{1}{504h^5}t^7 - \frac{152}{315}ht \\[2mm]
\beta_4(x) = \frac{61}{288h^2}t^4 + \frac{4}{63}ht - \frac{5}{24h}t^3 + \frac{11}{3720h^4}t^6 - \frac{1}{1008h^5}t^7 - \frac{41}{480h^3}t^5 \\[2mm]
\beta_5(x) = \frac{1}{5040h^5}t^7 - \frac{5}{144h^2}t^4 - \frac{4}{315}ht + \frac{1}{30h}t^3 + \frac{1}{3360h^4}t^6 + \frac{7}{480h^3}t^5
\end{array}\right] \tag{9}$$

Evaluating (9) at t=-1,-2,-3 and 5 with its first derivative evaluated at t=0,-1,-2,-3,-4 and 5 with t=(〚x〛 _n-x) and the results substituted in (7), the following discrete schemes were obtained

$$\begin{bmatrix}
1 & 0 & -\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{3}{4} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{5}{4} & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{4h} & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{4h} & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{4h} & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -\frac{1}{4h} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{4h} & 0 & 0 & 0 & & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+5} \\ y'_{n+1} \\ y'_{n+2} \\ y'_{n+3} \\ y'_{n+4} \\ y'_{n+5} \end{bmatrix}
=
\begin{bmatrix}
0 & \frac{3}{4} \\
0 & \frac{1}{2} \\
0 & \frac{1}{4} \\
4h & 1 \\
0 & -\frac{1}{4} \\
0 & -\frac{1}{4h} \\
0 & -\frac{1}{4h} \\
0 & -\frac{1}{4h} \\
0 & -\frac{1}{4h} \\
0 & -\frac{1}{4h}
\end{bmatrix}
\begin{bmatrix} y'_n \\ y_n \end{bmatrix}
+
\begin{bmatrix}
-\frac{337}{480}h^2 & -\frac{53}{120}h^2 & -\frac{71}{240}h^2 & -\frac{1}{240}h^2 & -\frac{1}{480}h^2 \\
-\frac{8}{15}h^2 & -\frac{13}{15}h^2 & -\frac{8}{15}h^2 & -\frac{1}{15}h^2 & 0 \\
-\frac{127}{480}h^2 & -\frac{29}{600}h^2 & -\frac{161}{240}h^2 & -\frac{1}{15}h^2 & \frac{1}{480}h^2 \\
\frac{1424}{315}h^2 & \frac{176}{315}h^2 & \frac{608}{315}h^2 & -\frac{16}{63}h^2 & \frac{16}{315}h^2 \\
\frac{23}{96}h^2 & \frac{13}{24}h^2 & \frac{11}{16}h^2 & \frac{15}{16}h^2 & \frac{7}{96}h^2 \\
-\frac{1403}{10080}h & -\frac{3497}{5040}h & -\frac{149}{1008}h & -\frac{571}{10080}h & \frac{61}{10080}h \\
\frac{191}{630}h & \frac{1}{63}h & -\frac{103}{315}h & -\frac{1}{315}h & -\frac{1}{630}h \\
\frac{481}{2016}h & \frac{2887}{5040}h & \frac{1159}{5040}h & -\frac{683}{10080}h & \frac{61}{10080}h \\
\frac{92}{315}h & \frac{124}{315}h & \frac{296}{315}h & \frac{118}{315}h & -\frac{4}{315}h \\
\frac{1733}{10080}h & \frac{3671}{5040}h & \frac{1943}{5040}h & \frac{2753}{2016}h & \frac{3197}{10080}h
\end{bmatrix}
\begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \end{bmatrix}
+
\begin{vmatrix}
-\frac{15}{240}h^2 \\
-\frac{1}{30}h^2 \\
-\frac{1}{60}h^2 \\
\frac{376}{315}h^2 \\
\frac{1}{48}h^2 \\
\frac{317}{10080}h \\
\frac{4}{315}h \\
\frac{41}{2016}h \\
\frac{4}{315}h \\
\frac{317}{10080}h
\end{vmatrix}
\left[f_n \right] \tag{9}$$

3. Basic properties of the method order and error constant

Expanding the discrete block method (9) in Taylor's series gives;

$$\sum_{j=0}^{\infty}\frac{(1)h^j}{j!}y_n^j - hy'_n - y_n - \frac{h^2}{1080}\sum_{j=0}^{\infty}\frac{h^j}{j!}y_n^{(j+2)}\begin{pmatrix}1231\,(0)^j - 4315\,(1)^j - 3044\,(2)^j\\ +1882\,(3)^j - 682\,(4)^j - 107\,(5)^j\end{pmatrix} = 0$$

$$\sum_{j=0}^{\infty}\frac{(2)h^j}{j!}y_n^j - 2hy'_n - y_n - \frac{h^2}{630}\sum_{j=0}^{\infty}\frac{h^j}{j!}y_n^{(j+2)}\begin{pmatrix}-355\,(0)^j - 1088\,(1)^j + 370\,(2)^j\\ -272\,(3)^j + 101\,(4)^j - 16(5)^j\end{pmatrix} = 0$$

$$\sum_{j=0}^{\infty}\frac{(3)h^j}{j!}y_n^j - 3hy'_n - y_n - \frac{h^2}{1120}\sum_{j=0}^{\infty}\frac{h^j}{j!}y_n^{(j+2)}\begin{pmatrix}-984\,(0)^j - 3501\,(1)^j + 72\,(2)^j\\ -870\,(3)^j + 288\,(4)^j - 45\,(5)^j\end{pmatrix} = 0$$

$$\sum_{j=0}^{\infty}\frac{(4)h^j}{j!}y_n^j - 4hy'_n - y_n - \frac{h^2}{315}\sum_{j=0}^{\infty}\frac{h^j}{j!}y_n^{(j+2)}\begin{pmatrix}-376\,(0)^j - 1424\,(1)^j - 176\,(2)^j\\ -608\,(3)^j + 80\,(4)^j - 16(5)^j\end{pmatrix} = 0$$

$$\sum_{j=0}^{\infty}\frac{(5)h^j}{j!}y_n^j - 5hy'_n - y_n - \frac{h^2}{10080}\sum_{j=0}^{\infty}\frac{h^j}{j!}y_n^{(j+2)}\begin{pmatrix}-15250\,(0)^j - 59375\,(1)^j - 12500\,(2)^j\\ -31250\,(3)^j - 6250\,(4)^j - 1375\,(5)^j\end{pmatrix} = 0$$

$$\sum_{j=0}^{\infty}\frac{(1)h^j}{j!}y_n^{(j+1)} - y'_n - \frac{h}{1440}\sum_{j=0}^{\infty}\frac{h^j}{j!}y_n^{(j+2)}\begin{pmatrix}-475\,(0)^j - 1427\,(1)^j + 798\,(2)^j\\ -482\,(3)^j + 173\,(4)^j - 27\,(5)^j\end{pmatrix} = 0$$

$$\sum_{j=0}^{\infty}\frac{(2)h^j}{j!}y_n^{(j+1)} - y'_n - \frac{h}{90}\sum_{j=0}^{\infty}\frac{h^j}{j!}y_n^{(j+2)}\begin{pmatrix}-28\,(0)^j - 129\,(1)^j - 14\,(2)^j\\ -14\,(3)^j + 4(4)^j - (5)^j\end{pmatrix} = 0$$

$$\sum_{j=0}^{\infty}\frac{(3)h^j}{j!}y_n^j - y'_n - \frac{h}{160}\sum_{j=0}^{\infty}\frac{h^j}{j!}y_n^{(j+2)}\begin{pmatrix}-51(0)^j - 219\,(1)^j - 114\,(2)^j\\ -114\,(3)^j + 21(4)^j - 3(5)^j\end{pmatrix} = 0$$

$$\sum_{j=0}^{\infty}\frac{(4)h^j}{j!}y_n^j - y'_n - \frac{h}{45}\sum_{j=0}^{\infty}\frac{h^j}{j!}y_n^{(j+2)}\begin{pmatrix}1-14\,(0)^j - 64\,(1)^j - 24\,(2)^j\\ -64\,(3)^j - 14\,(4)^j\end{pmatrix} = 0$$

$$\sum_{j=0}^{\infty}\frac{(5)h^j}{j!}y_n^j - y'_n - \frac{h}{288}\sum_{j=0}^{\infty}\frac{h^j}{j!}y_n^{(j+2)}\begin{pmatrix}-95\,(0)^j - 375\,(1)^j - 250\,(2)^j\\ -250\,(3)^j - 375\,(4)^j - 95\,(5)^j\end{pmatrix} = 0$$

Collecting like terms in powers of h, leads to the following expression; $\bar{C}_0 = \bar{C}_1 = \ldots = \bar{C}_6 = (0,0,0,0,0,0,0,0,0,0)^T$

and $C_7 = \left(-\frac{29}{2240}, -\frac{8}{945}, -\frac{275}{12095}, -\frac{199}{24192}, -\frac{19}{945}, -\frac{141}{4480}, -\frac{8}{189}, -\frac{1375}{24192}, -\frac{863}{60480}, -\frac{37}{3780}\right)^T$

Hence the block method has order of p= 〚(5,5,5,5,5,5,5,5,5,5)〛 ^T and with error constants of

Consistency

Following [5] and [8] the block method is consistent since it has orders p=5>1

Zero stability

The block solution of (9) is said to be zero stable if the roots z_r; r =1,…,n of the first characteristic polynomial ρ̌(z), defined by

p(z)=det|zQ-T|

satisfies |z_r |≤1 and every root with |z_r |=1 has multiplicity not exceeding power of the differential equation in the limit as h→0. From the block solution (11.0), we have

$$p(z) = z^8(z^2 - 1)$$

This shows that our method is zero stable, since every root with |z_r |=1 has multiplicity not exceeding power of the differential equation in the limit as h→0.

Convergence

According to [5], [6] and [7], the block method is convergent since it is consistent and zero stable

4. Numerical Experiments

In this section, we implement the proposed method to solve two second order initial value problems of ordinary differential equations and examine the efficiency and accuracy of the proposed block method. The absolute errors of the test problems in [1] and [12] which are both of order six are compared with the proposed method.

Example 1

Consider the IVP $y'' - 100y = 0, y(0) = 1, y'(0) = -10, h = 0.01$

With exact solution $y(x) = e^{-10x}$ as with the results shown in Table 1

Example 2

Consider the IVP $y'' - x(y')^2 = 0, y(0) = 1, y'(0) = \frac{1}{2}, h = \frac{1}{30}$

With exact solution as $y(x) = 1 + \frac{1}{2}\ln\left(\frac{2+x}{2-x}\right)$ with the results shown in Table 2

Example 3

Consider the IVP in Jator and Li (2009) $y'' - 4y' + 8y = x^3, y(0) = 2, y'(0) = 4, x \in [0,1]$

With exact solution $y(x) = e^{2x}\left(2\cos 2x - \frac{3}{64}\sin 2x\right) + \frac{3x}{32} + \frac{3x^2}{16} + \frac{x^3}{8}$ with the results shown in Table 3

Table 1: Comparing the error in proposed results with the error in [1]

x	Exact solution	Result of proposed method	Error in proposed method	Error in [1]
0.01	0.9048374180359596	0.9048374181022410	6.62814×10^{-11}	1.350×10^{-7}
0.02	0.8187307530779819	0.8187307532407858	1.62804×10^{-10}	3.660×10^{-7}
0.03	0.7408182206817179	0.7408182209384693	2.56751×10^{-10}	6.050×10^{-7}
0.04	0.6703200460356393	0.6703200463851016	3.49462×10^{-10}	8.500×10^{-7}
0.05	0.6065306597126334	0.6065306601858334	4.73200×10^{-10}	1.100×10^{-6}
0.06	0.5488116360940264	0.5488116367982202	7.04194×10^{-10}	1.370×10^{-6}
0.07	0.4965853037914095	0.4965853047515867	9.60177×10^{-10}	1.450×10^{-6}
0.08	0.4493289641172216	0.4493289653404401	1.22322×10^{-9}	1.600×10^{-6}
0.09	0.4065696597405991	0.4065696612367939	1.49619×10^{-9}	1.760×10^{-6}
0.10	0.3678794411714423	0.3678794429722842	1.80048×10^{-9}	1.950×10^{-6}
0.11	0.3328710836980796	0.3328710858837756	2.18570×10^{-9}	2.100×10^{-6}
0.12	0.3011942119122021	0.3011942145155064	2.60330×10^{-9}	2.370×10^{-6}

Table 2: Comparing the error in proposed results with the error in [12]

x	Exact solution	Result of proposed method	Error in proposed method	Error in [12]
0.1	1.050041729278491	1.050041729278774	2.8300000×10^{-13}	5.89100000×10^{-6}
0.2	1.100335347731076	1.100335347732127	1.0510000×10^{-12}	8.23990000×10^{-5}
0.3	1.151140435936467	1.151140435939479	3.0120000×10^{-12}	3.46421000×10^{-4}
0.4	1.202732554054082	1.202732554062001	7.9190000×10^{-12}	7.52010000×10^{-4}
0.5	1.255412811882995	1.255412811897937	1.4942000×10^{-11}	1.38028300×10^{-4}

Table 3: Comparing the error in proposed results with the error in [15]

x	Exact solution	Result of proposed method	Error in proposed method p = 5	Error in [15] p = 5
0.1	2.394112576996396	2.394113776778579	1.19978×10^{-6}	5.10704×10^{-6}
0.2	2.748141332426423	2.748144902927067	3.57051×10^{-6}	1.49586×10^{-5}
0.3	3.007866940511069	3.007873568711235	6.62820×10^{-6}	2.78532×10^{-5}

0.4	3.101762405774209	3.101772928904299	1.05231×10^{-5}	4.28908×10^{-5}
0.5	2.939543100745261	2.939558771571024	1.56708×10^{-5}	6.70307×10^{-5}
0.6	2.411836534415715	2.411859807644079	2.32732×10^{-5}	1.02637×10^{-4}
0.7	1.391554830489843 -	1.391587372126977 -	3.25416×10^{-5}	1.44907×10^{-4}
0.8	0.2623267583343576 -	0.2622836072150302 -	3.25416×10^{-5}	1.90905×10^{-4}
0.9	2.697771160773071 -	2.697716380090280 -	5.47807×10^{-5}	2.39733×10^{-4}
1.0	6.058560720845667	6.058494007435692	6.67134×10^{-5}	2.94670×10^{-4}

5. Conclusion

The desirable property of a numerical solution is to behave like the theoretical solution of the problem which can be seen in the above result. In this paper, it has been shown that continuous collocation methods for solving ordinary differential equations can equally be derived through the approach in this study. In this study, a new block method (k=5) that is convergent and absolutely stable is presented. The proposed method is used to solve numerically linear as well as nonlinear initial value problems of ordinary differential equations of the second order. The results of the presented examples revealed that our method performed relatively better than those presented in [1], [12] and [15] respectively.

6. References

1. Awari YS, Chima EE, Kamoh NM, Oladele FL. A Family of Implicit Uniformly Accurate Order Block Integrators for the Solution of Second Order Differential Equations. International Journal of Mathematics and Statistics Invention (IJMSI). 2014; 2:34-46
2. Awoyemi DO. A Class of Continuous Linear Multistep Method for General Second Order Initial Value Problems in Ordinary Differential Equations. International Journal of Computer Mathematics. 1991; 72:29-37.
3. Awoyemi DO. A New Sixth-Order Algorithm for General Second Order Ordinary Differential Equation. International Journal of computational mathematics. 2001; 77:117-124.
4. Awoyemi DO, Kayode SJ. An Implicit Collocation Method for Direct Solution of Second Order ODEs. Journal of Nigeria Association of Mathematical Physics. 2005; 24:70-78.
5. Fatunla SO. Block Method for Second Order Ordinary Differential Equations. International Journal of Computer Mathematics. 1991; 41(1&2):55-63.
6. Henrici P. Discrete Variable Methods in Ordinary Differential Equations. New York: John Willey and Sons, 1962.
7. Lambert JD. Computational Methods in Ordinary Differential Equations. New York: John Wiley and Sons, 1973.
8. Lambert JD. Numerical Methods for Ordinary Differential Equations. New York: John Wiley and Sons, 1991.
9. Okunuga SA, Ohigie J. A New Derivation of Continuous Multistep Methods Using Power Series as Basis Function. Journal of Modern Mathematics and Statistics. 2009; 3(2):43-50.
10. Kuboye JO, Omar Z. Numerical Solution of Third Order Ordinary Differential Equations Using a Seven-Step Block Method. International Journal of Mathematical Analysis. 2015; 9(15):743-754.
11. Yahaya YA, Badmus AM. A Class of Collocation Methods for General Second Order Ordinary Differential Equations, African Journal of Mathematics and Computer Science Research. 2009a; 2(04):069-072.
12. Yakusak NS, Emmanuel S, Ogunniran MO. Uniform Order Legendre Approach for Continuous Hybrid Block Methods for the Solution of First Order Ordinary Differential Equations. IOSR Journal of Mathematics. 2015; 11(1):09-14.
13. Owolabi KM. A Rebust Optimal Order Formula for Direct Integration of Second Order Orbital Problems. Advances in Physics Theories and Applications. 2015; 42:21-26.
14. Zill DG, Warren WS. Differential Equations with Boundary- Value Problem; Eight edition books/Cole, Cengage Learning, 2013.
15. Jator S, Li N. A Self-Starting Linear Multistep Method for the Direct Solution of General Second Order Initial Value Problem. International Journal of Computer Mathematics. 2009; 86(5):827-836.
16. Jator SN. A Sixth Order Linear Multistep Method for the Direct Solution of y^(//) = f(x,y,y^/) International Journal of Pure and Applied Mathematics. 2007; 40(4):457-472.
17. Fatunla SO. Block methods for second order IVPs, International Journal Compter Mathematics, 1991, 41.
18. Awoyemi DO, Kayode SJ. An Implicit Collocation Method for Direct Solution of Second Order Ordinary Differential Equations. Journal of the Nigerian Mathematical Society, 2005b; 24:70-78.
19. Adeniyi RB, Alabi MO. Continuous Formulation of a Class of Accurate Implicit LMMs with Chebyshev Basis Function in a Collocation Technique. Journal of the Mathematical Association of Nigeria (Abacus), 2007; 34(2A):58-77.
20. Majid ZA, Azimi NA, Suleiman M. Solving Second Order Ordinary Differential Equations Using Two Point Four Step Direct Implicit Block Method. European Journal of Scientific Research. 2009; 31(1):29-36.